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GIMRADA Research Note No. 4

INVESTIGATIONS OF BASIC GEOMETRIC QUALITY OF AERIAL PHOTOGRAPHS AND SOME RELATED PROBLEMS

By K. Bertil P. Hallert

17 August 1962

U.S. ARMY
GEODESY, INTELLIGENCE AND MAPPING
RESEARCH AND DEVELOPMENT AGENCY
CORPS OF ENGINEERS

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CORRECTION SHEET

page v	line 11	Read correlation effects
page 4	line 7	Read inversely proportional to
page 13	line 4	$c = 152$
	line 8	Read N54 for Ntr
	line 15	insert -56 in last column
page 19	line 3 from bottom read $(9N94-8[dy])^2$	
page 22	line 11	Read inversely proportional
page 35	line 4 from bottom Read c = 152 mm.	
page 54	upper left corner of diagram, change 008 to 308	
page 76	Table XVI dx_o and dy_o units are in (microns)	
	$d\kappa$, $d\varphi$, $d\omega$ are measured in cc (centesimal seconds)	

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The Director
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Prepared by

K. Bertil P. Hallert
Research and Analysis Division
U. S. Army Engineer
Geodesy, Intelligence and Mapping Research and Development Agency
Fort Belvoir, Virginia

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CONTENTS

SUMMARY

This paper covers a series of tests of the basic geometrical quality of photographs obtained from aerial camera calibrations in a multicollimator and from aerial photography obtained under operational conditions. The elements of the interior orientation and the most important regular (systematic) errors of the image coordinates are determined as parameters in least squares adjustments, and the remaining irregular errors of the image coordinates are estimated as standard errors of unit weight of the image coordinates. In the multicollimator tests, film and glass plate negatives were tested and compared concerning the geometrical quality. A criterion for tangential distortion is developed, and some correlation effect between residual image coordinate errors is studied.

A variation of the geometrical quality (weight) of image coordinates was found, and some possible reasons for this variation are more closely investigated. The resolving power, the thickness variations of films, and the locations of developed grains (details) within the emulsion were preliminarily studied. Some tests of diapositive printers were made.

This paper concludes that:

a. The method of least squares is of great value for the calibration of aerial cameras in multicollimators and for additional tests of aerial photographs after photography of regular test fields.

b. The basic geometrical quality, expressed as standard errors, of the elements of the interior orientation, of regular errors of the image coordinates, and of other functions of the image coordinate measurements can be expressed uniquely in terms of the standard error of unit weight of the image coordinates.

c. The basic geometrical quality of image coordinates in glass plates and in films have proved to be approximately identical. The standard error of unit weight of the image coordinates increases with the radius from the principal point. The average was found to be about 3 to 4 microns.

d. Among the reasons for the weight variations of the image coordinates are variations of the resolving power, lacking flatness of the image plane (caused by variations of the thickness of films and emulsions among other things), and varying location of the details within the emulsion.

e. More attention should be paid to the basic physical qualities of the image material.

f. Grid test fields should be arranged in the terrain for tests of the basic geometrical qualities of vertical and oblique photographs from different flying altitudes.

INVESTIGATIONS OF BASIC GEOMETRIC QUALITY OF AERIAL PHOTOGRAPHS AND SOME RELATED PROBLEMS

I. INTRODUCTION

The quality of a photograph can be defined with respect to the intended use. If the primary application of the photograph is the interpretation of the contrasts and contrast differences, the interpretative quality in particular connected with resolving power, etc., is of basic importance. However, if the use of the photograph or photographs shall serve the purpose of determining geometric qualities such as size, form, and position of the photographed object, the geometrical quality of the photograph is of fundamental importance. In fact, the geometrical quality of the photograph must, together with the influences from the measuring instruments and the operator, determine the quality of the final product of the photogrammetric procedure. Therefore, it is most important to determine the basic geometrical quality of the photographs for all photogrammetric activity and to express this quality in a well-defined and clear way. Some experiments performed for this purpose are described below, and some applications of the results of the experiments to actual photogrammetric problems are treated.

There is evidently a certain connection between the interpretative and the geometrical quality of the photograph. In fact, all measurements in a photograph must be preceded by an interpretation of the detail to be measured; for many applications of photo interpretation, the geometrical qualities of the actual objects are of great importance. Further, there may be a close relation between certain concepts of the interpretative quality (resolving power, sharpness, acuity, etc.) and the geometrical quality of image coordinates. However, the primary purpose of this paper is to show the results of certain research concerning the geometrical quality of photographs and some applications of the results.

II. INVESTIGATION

1. Definition of Geometrical Quality of a Photograph. In general, a photograph shall be a mathematically correct plane central perspective of the photographed object. This means that all points in the object shall be geometrically imaged by rays or straight lines which all pass one point, the perspective center. The elements of the interior orientation of a photograph shall primarily define the position of the perspective center with respect to the coordinate system of the photograph, which usually is physically defined by fiducial marks. This relative position is usually

defined by the principal point and the principal distance. However, since there are always two perspective centers in a lens and since it is not possible to make the physical realization exactly coincide with the mathematical concept, there must also be information available on the systematic (regular) disturbances of the image coordinates with respect to the ideal, mathematical positions as determined by the strict central projection and usually described as ideal image coordinates. Such regular errors of the image coordinates are primarily radial and tangential distortion. Further regular errors of the image coordinates are caused by affine deformations of the image, primarily caused by different shrinkage in different directions. Even if these regular errors are determined, however, the image coordinates will always be affected by errors which individually do not follow any specific law concerning magnitude and direction and which, therefore, are denoted irregular errors. These errors may have many causes as for instance residuals after the determination of the radial and tangential distortion and the affine deformation, irregular shrinkage, blur, the errors of the measurements of the coordinates due to the settings and readings of the scales, etc. According to the central limit theorem, the irregular errors may be expected to be at least approximately normally distributed; this is an important condition for the correct treatment of such errors, in particular their propagation.

In summary, the geometrical quality of a photograph is here defined by the regular (systematic) errors and the irregular errors which affect the image coordinates for a given set of values of the principal point and the principal distance. The principal point is assumed to be defined in the photograph by the intersection of fiducial lines, the principal distance by a number referring to the distance between the principal point and the interior perspective center. The radial distortion is defined by a radial distortion curve for a certain principal distance (camera constant, calibrated focal length), the tangential distortion by the deformation of straight lines in the image through the principal point, and the affine deformation by the scale factor differences in two orthogonal directions in the image. Finally, the irregular errors are expressed statistically as the standard error of unit weight of the image coordinates after an adjustment according to the method of least squares with a well-defined set of parameters.

Evidently, all these data must refer to the photograph as obtained through real photography under operational conditions. Therefore, tests under such conditions are most desirable. Since there may be factors that influence the geometrical quality of the images differently depending upon the outer circumstances, it is necessary to test photographs repeatedly and under different conditions concerning for instance the flying altitude in aerial photography. On the other hand, it is also of great interest to

investigate the geometrical quality of the image coordinates in connection with the calibration of the camera in order to determine possible differences in the geometrical quality of the photographs from the stage of calibration to the real photography. The regular and irregular errors from the calibration procedure may be of particular interest for the judgment of the quality of the lens while the corresponding errors from aerial photography are of the greatest interest for the entire procedure and for the establishment of specifications and tolerances. It is also quite evident that good results of tests of photographs obtained under operational conditions necessarily mean that the quality of the lens is satisfactory. Good results from laboratory lens testing do not guarantee the quality of the photographs which are used for the photogrammetric measurements and, consequently, are deciding for the quality of the final results.

The results of performed tests of aerial and terrestrial photographs will be briefly summarized below before a description of some recent tests of photographs from a multicollimator procedure for camera calibration. Further, some recent tests of basic qualities of negative material, in particular films, will be described and some applications of the results of detailed geometrical tests of aerial photographs will be discussed.

2. Quality Tests of Aerial Photographs. In two publications (references 1 and 2), the results of a series of tests of aerial photographs have been presented. A test field in the terrain, consisting of a set of very accurately surveyed and signalled points in the corners of a big grid, was photographed from the air with the aerial cameras to be tested. The image coordinates of the points were measured and compared with the ideal image coordinates of the same points if a strict central perspective had been available. The discrepancies which necessarily must occur were interpreted as caused by regular and irregular sources of errors. The possible regular sources of errors are primarily the approximations of the elements of the external orientation of the photographs, which were used for the computations of the ideal image coordinates, the radial distortion, the tangential distortion, and the affine deformations. For the determination of the radial distortion, the adjustment of the discrepancies was made separately for each set of points which are located on a circle around the principal point. The affine deformations were expressed as different scale factors in different orthogonal directions in the photographs or as different principal distances or flying altitudes. If the first approximations of the elements of the external orientation were too large, an iterative procedure was applied. Further, corrections were applied for the elevation differences of the terrain points. Through these investigations, the radial distortion effects of the aerial photographs and the affine deformations were determined and, furthermore, the irregular errors were estimated for each circle as a standard error of

unit weight from the adjustment according to the method of least squares which was applied throughout. A closer investigation of these standard errors proved a considerable regular variation from circle to circle. This means that there is a considerable variation of the geometrical quality of the image coordinates within the image. In publication (2), this variation has been computed and expressed in terms of weights, which are defined as inverse proportional to the squares of the standard errors of unit weight. If the weight unity is chosen for the principal point of the photograph, the points in the corners of the photographs proved to have weights of the order of magnitude 1/20. Similar results have also been obtained for terrestrial photographs as treated in publication (3). For each individual terrain point, the residual image coordinate errors after the adjustment were computed and graphically presented (see (1) and (2)). The residuals were referred to the adjustment of one of the circles containing eight points in addition to the center point. The radial distortion and the affine deformations were also corrected in the residuals which consequently can be regarded to be of mainly irregular nature. The possible tangential distortion is included, however, and one of the problems to be treated below is to find a method to determine possible tangential distortion from the residual coordinate errors and to establish a general criterion for this purpose. A further problem to be treated is the correlation between the residual errors within the photograph. It is evident that this correlation must become stronger the closer the points are located since the shrinkage of the films and emulsion certainly is one of the most important reasons for the correlation. Therefore, investigations of the correlation between the residual errors in the corners of the squares of the grid may give important information about, for instance, how dense a grid in the camera should be for the correction of systematic image coordinate errors. There are several other problems the solutions of which may be facilitated by statistical information about the correlation between image coordinate errors.

Aerial cameras have also been tested from high towers through photography of test fields on the ground. These experiments were mainly made in order to find the specific influence of certain sources of error upon the image coordinates as, for instance, varying temperature, different types of filters, vibrations, and different types of film in comparison with glass plates. From the residual errors of the image coordinates, detailed investigations of many interesting problems can be performed in a similar way as has been indicated above. So far, investigations of the weight variation of the image coordinates have been performed and have proved similar variations as were found from the tests of the image coordinates in aerial photographs from high altitudes.

3. Quality Tests of Image Coordinates from Calibration of Aerial Cameras with Multicollimators. In the calibration of aerial cameras with the multicollimator procedure, a set of targets is imaged on the negative of the camera through collimators which are directed through the outer perspective center of the camera lens. The collimators are arranged in two orthogonal planes which also contain the diagonals of the image. In addition, there are also some collimators between the diagonals (see (4)).

Since the collimators are adjusted in certain angles, there are evidently several perspective conditions which can be used for determinations and adjustments of discrepancies and which will give very good information about the accuracy of the image coordinates of the negative. The procedure is identical with the test of aerial photographs after photography from the air as described in paragraph 2 and in the references. From the angles between the collimators and an approximate principal distance, the "given" image coordinates are computed and then compared with the corresponding measured image coordinates from the photographs taken in connection with the calibration. The discrepancies are then interpreted as caused by the approximative principal distance, the approximations in the remaining five elements of the external or internal elements of orientation, radial distortion, and affine deformations. The adjustment can be performed according to the formula systems derived in the publication (5).

For practical application of these principles, a set of photographs was received from the U. S. Geological Survey. One photograph was on glass plate, and five were on film. The adjustment experiments have been applied to the glass plate and to two of the films selected at random from the strip. Later, another set of glass plates and films from a second camera was received and tested.

a. Determination of Given Image Coordinates. The positions of the targets in the image plane and the notations and the angles between the central collimator and the surrounding collimators are shown in Fig. 1. In Table I, the computation of the given image coordinates is shown. From preliminary computations, two different principal distances were found, one for the plate = 152.079 mm and one for the films = 151.941 mm. With these two data and the given angles between the collimators and between the planes of the collimators, the coordinates of Table I have been computed as easily can be seen from the figures of the table. In other cases of tests of glass plate or film negatives, slight changes of the given coordinates might be necessary due to shrinkage and other causes. In order to avoid too large figures in the computations, it is always advisable to use such a preliminary principal distance that the discrepancies between measured and given coordinates become as small as possible.

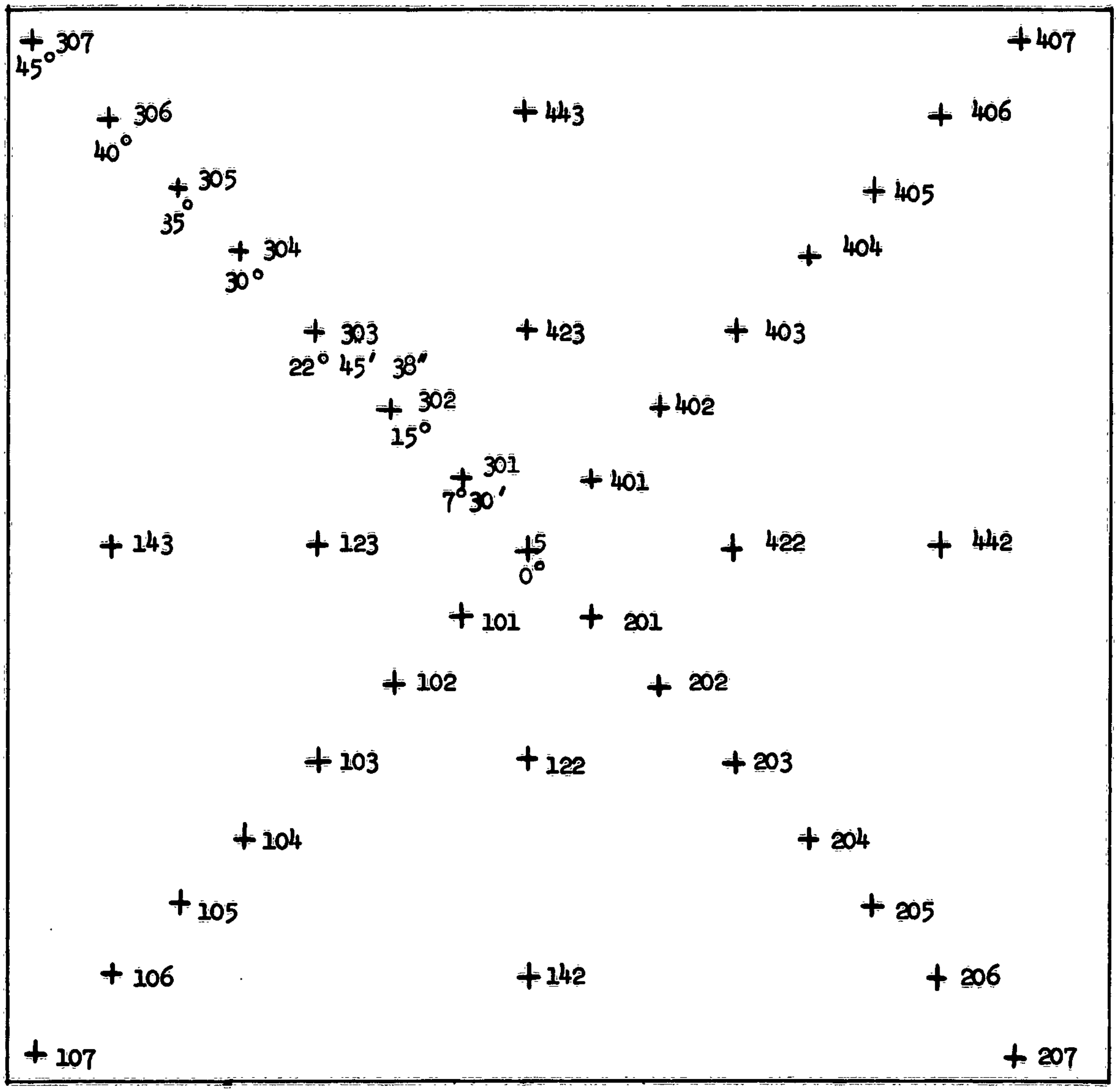


Fig. 1. Positions and notations of the targets in the image plane. The angles between the central collimator and all other collimators are also shown, cf Table I.

Table I. Determination of Given Coordinates of Image Points

Angles α		tan α	151.941tanα (mm)	x_{film} (mm)	y_{film} (mm)	152.079tanα	x_{plate} (mm)	y_{plate} (mm)
Sexages.	Centes.							
$7^{o}30'$	$8^{g}3333$	0.131652	20.003	14.144	14.144	20.022	14.158	14.158
$15^{o}00'$	$16^{g}6667$	0.267950	40.713	28.788	28.788	40.750	28.815	28.815
$16^{o}31'26''$	$18^{g}3599$	0.296667	45.076					
$22^{o}45'38''$	$25^{g}2895$	0.419552	63.747	45.076	45.076	63.805	45.117	45.117
30^{o}	$33^{g}3333$	0.577349	87.723	62.030	62.030	87.803	62.086	62.086
$30^{o}40'54''$	$34^{g}0907$	0.593323	90.150					
35^{o}	$38^{g}8889$	0.700208	106.390	75.229	75.229	106.487	75.298	75.298
40^{o}	44.4444	0.839098	127.493	90.151	90.151	127.609	90.233	90.233
45^{o}	50.0000	1.000000	151.941	107.439	107.439	152.079	107.536	107.536

The image coordinates according to Table I obtain their signs according to the location of the points in the image.

b. Measurements of Image Coordinates and Computations.

(1) Adjustment, Assuming One Scale Factor of Image Coordinates. The image coordinates of the plate and of the films were measured in a Mann comparator after approximative orientation of the images in the instrument. All measurements were repeated twice independently, and in opposite sequence. The precision of the settings was separately determined from 20 repeated settings, and the average standard deviation of one measurement was found to be 2.4 microns. This value is larger than was obtained from grid measurements and is probably influenced by the photographic quality of the targets. From the two independent series of measurements in each image, the root mean square value of the differences was found to be 3.3 microns. Since each difference consists of two settings, the standard deviation on one setting can be computed from the root mean square value in dividing by 1.414. We then find 2.2 microns which agrees well with the standard deviation from repeated settings in one point only. In connection with the measurements of image coordinates, the coordinates of fiducial marks were also measured. Usually, the image coordinates of these points became determined with less precision than the coordinates of the collimator targets due to the lower definition of the fiducial marks.

The results of the measurements in the images were next translated to a coordinate system, the origin of which was located in the center point 5. This translation was made in subtracting the measured coordinates of point 5 from all the other coordinates.

The translated coordinates were then directly compared with the given coordinates of the image points as defined by the preliminary principal distance and the angles between the center collimator and the other collimators according to Fig. 1. The discrepancies between the measured and the given coordinates were then determined as errors of the measured coordinates and were consequently defined as $dx = x_{measured} - x_{given}$ and $dy = y_{measured} - y_{given}$. A complete example of these computations is shown in Table II. The errors dx and dy of the measured coordinates were next interpreted as caused by the six elements of the exterior orientation of the camera in relation to the multicollimator system. One of the elements, the scale of the photograph, however, was interpreted as depending upon the principal distance of the camera, which was chosen rather arbitrarily in connection with the computation of the given coordinates. Since the further computations were made for point combinations into circles around the center point 5, systematic variations of the scale in relation to the different radii caused variations in the principal distance from circle to circle. This variation was then interpreted as caused by radial distortion. As will be shown by the formula systems below, the radial distortion can be obtained and expressed very conveniently directly from the errors of the image coordinates after using the above mentioned relation between the scale variation and the variation of the principal distance.

The general procedure for the computations is entirely founded upon the method of least squares and has been published earlier by the author. Therefore, only a brief summary will be given here in order to explain the background of the scheme to be used for the practical calculations. Referring to publications 6 and 7 where the entire procedure is described in details, we start with the basic correction equations for the errors dx and dy as functions of the elements of orientation of the photographs:

$$v_x = - dx_o - \frac{x}{c}dc + y d\kappa + \left(1 + \frac{x^2}{c^2}\right)c d\varphi - \frac{xy}{c} d\omega - dx \qquad (1)$$

$$v_y = - dy_o - \frac{y}{c}dc - x d\kappa + \frac{xy}{c}d\varphi - \left(1 + \frac{y^2}{c^2}\right)c d\omega - dy \qquad (2)$$

$$dx = x_{measured} - x_{given} \qquad dy = y_{measured} - y_{given}$$

If affine deformations are present, for instance due to different shrinkage in the x- and y-directions, two different expressions are introduced for dc -- dc_x and dc_y. Also other possible systematic errors can be introduced in this basic differential expression. In each combination, the measured coordinates in five points are to be taken into account. Since there are six unknowns in the expressions above, there are four redundant observations to be adjusted. In case

Table II. Measured and Transformed Image Coordinates, Given Coordinates and Discrepancies (c (preliminary) = 152.188 mm Film II)

Point	Measured Coord.		Transl. Coord.		Given Coord.		Discrepancies	
	x (mm)	y (mm)	x (mm)	y (mm)	x (mm)	y (mm)	dx (microns)	dy (microns)
5	178.072	241.222	0.000	0.000	0.000	0.000		
101	163.896	227.051	-14.176	-14.171	-14.168	-14.168	- 8	- 3
102	149.232	212.383	-28.840	-28.839	-28.836	-28.836	- 4	- 3
103	132.912	196.061	-45.160	-45.161	-45.149	-45.149	-11	-12
104	115.934	179.088	-62.138	-62.134	-62.131	-62.131	- 7	- 3
105	102.711	165.873	-75.361	-75.349	-75.352	-75.352	- 9	+ 3
106	087.770	150.928	-90.301	-90.294	-90.298	-90.298	- 3	+ 4
1	071.516	135.870	-106.556	-105.352				
142	178.066	150.910	-0.006	-90.312	0.000	-90.298	- 6	-14
2	283.420	134.645	+105.348	-106.577				
201	192.243	227.045	+14.171	-14.177	+14.168	-14.168	+ 3	- 9
202	206.912	212.375	+28.840	-28.847	+28.836	-28.836	+ 4	-11
203	223.220	196.057	+45.148	-45.165	+45.149	-45.149	- 1	-16
204	240.196	179.083	+62.124	-62.139	+62.131	-62.131	- 7	- 8
205	253.408	165.871	+75.336	-75.351	+75.352	-75.352	-16	+ 1
206	268.356	150.917	+90.284	-90.305	+90.298	-90.298	-14	- 7
301	163.900	255.401	-14.172	+14.179	-14.168	+14.168	- 4	+11
302	149.229	270.072	-28.843	+28.850	-28.836	+28.836	- 7	+14
303	132.914	286.389	-45.158	+45.167	-45.149	+45.149	- 9	+18
304	115.942	303.368	-62.130	+62.146	-62.131	+62.131	+ 1	+15
305	102.728	316.586	-75.344	+75.364	-75.352	+75.352	+ 8	+12
306	087.785	331.530	-90.287	+90.308	-90.298	+90.298	+11	+10
3	072.736	347.787	-105.336	+106.565				
4	284.650	346.559	+106.578	+105.337				
143	087.766	241.231	-90.306	+0.009	-90.298	0.000	- 8	+ 9
123	132.907	241.225	-45.165	+0.003	-45.149	0.000	-16	+ 3
423	178.072	286.390	0.000	+45.168	0.000	+45.149	0	+19
443	178.076	331.544	+0.004	+90.322	0.000	+90.298	+ 4	+24
401	192.245	255.399	+14.173	+14.177	+14.168	+14.168	+ 5	+ 9
402	206.916	270.064	+28.844	+28.842	+28.836	+28.836	+ 8	+ 6
403	223.228	286.381	+45.156	+45.159	+45.149	+45.149	+ 7	+10
404	240.207	303.359	+62.135	+62.137	+62.131	+62.131	+ 4	+ 6
405	253.426	316.578	+75.354	+75.356	+75.352	+75.352	+ 2	+ 4
406	268.372	331.521	+90.300	+90.299	+90.298	+90.298	+ 2	+ 1
122	178.068	196.053	-0.004	-45.169	0.000	-45.149	- 4	-20
422	223.232	241.218	+45.160	-0.004	+45.149	0.000	+11	- 4
442	268.376	241.219	+90.304	-00.003	+90.298	0.000	+ 6	- 3

Note: Points 1, 2, 3, and 4 are fiducial marks. Given distances 212.000 mm.

of affine adjustment, there are seven unknowns and, consequently, three redundant observations. According to the method of least squares, the corrections of the preliminary elements of orientation shall be determined under the condition that the sum of the squares of the v_x and the v_y becomes a minimum. This leads to the normal equations which are to be formed together with the equations for the determinations of the sum of the squares $[v_x v_x + v_y v_y]$ and the weight and correlation numbers. Since this procedure ought to be very well known, only the results of the solutions of the normal equations will be shown here (see (8)). For a given circle with the radius r, the corrections of the elements of orientation are as follows:

$$dx_o = -\frac{r^2 + c^2}{3r^2}[dx] + \frac{2r^2 + 5c^2}{12r^2} N_{53} \; ; \tag{3}$$

$$dy_o = -\frac{r^2 + c^2}{3r^2}[dy] + \frac{2r^2 + 5c^2}{12r^2} N_{54} \; ; \tag{4}$$

$$dc = -\frac{c\sqrt{2}}{8r} N_{51} \tag{5}$$

$$d\varkappa = -\frac{\sqrt{2}}{8r} N_{52} \tag{6}$$

$$d\varphi = \frac{c}{12r^2}(5\,N_{53} - 4[dx]) \tag{7}$$

$$d\omega = \frac{c}{12r^2}(4\,[dy] - 5\,N_{54}) \tag{8}$$

$$\text{Radial distortion:} \quad dr' = \frac{\sqrt{2}}{8} N_{51} \tag{9}$$

The symbols are defined as follows:

$[dx] = dx_1 + dx_2 + dx_3 + dx_4 + dx_5$

$[dy] = dy_1 + dy_2 + dy_3 + dy_4 + dy_5$

$N_{51} = - dx_1 + dx_2 - dx_3 + dx_4 - dy_1 - dy_2 + dy_3 + dy_4$

$N_{52} = + dx_1 + dx_2 - dx_3 - dx_4 - dy_1 + dy_2 - dy_3 + dy_4$

$$N_{53} = dx_1 + dx_2 + dx_3 + dx_4 + dy_1 - dy_2 - dy_3 + dy_4$$

$$N_{54} = + dx_1 - dx_2 - dx_3 + dx_4 + dy_1 + dy_2 + dy_3 + dy_4 \tag{10}$$

$$[vv] = [dxdx] + [dydy] - \frac{[dx]^2 + [dy]^2}{5} - \frac{N_{51}^2 + N_{52}^2}{8} -$$

$$- \frac{(5N_{53} - 4[dx])^2 + (5N_{54} - 4[dy])^2}{120} \tag{11}$$

The standard error of unit weight:

$$s_o = \sqrt{\frac{[vv]}{10-6}} = \frac{1}{2}\sqrt{[vv]} \tag{12}$$

The standard error of the standard error of unit weight:

$$s_{s_o} = 0.35 s_o \tag{13}$$

The confidence limits of the standard error of unit weight:

on the 5-percent level: $0.6s_o - 2.9s_o$

on tne 1-percent level: $0.5s_o - 4.4s_o$ (14)

The weight and correlation numbers:

$$Q_{x_o x_o} = Q_{y_o y_o} = \frac{2r^4 + 5c^4 + 4r^2c^2}{6r^4}$$

$$Q_{cc} = \frac{c^2}{4r^2}$$

$$Q_{\varkappa\varkappa} = \frac{1}{4r^2}$$

$$Q_{\varphi\varphi} = Q_{\omega\omega} = \frac{5c^2}{6r^4}$$

$$Q_{x_o\varphi} = - Q_{y_o\omega} = \frac{c}{6r^4}(2r^2 + 5c^2)$$

$$Q_{drdr} = \frac{1}{4} \tag{15}$$

In order to facilitate the practical calculations, forms were made in which the basic data conveniently could be transformed into corrections, radial distortion, and standard error of unit weight. Such a form is shown in Table III. An example taken from Table II (circle 6) is completely treated in the scheme (Table III). In this way, the two above-mentioned sets of test photographs from the multicollimator instrument were treated. In the first series, one glass plate and two films were completely measured and the results were computed and will be shown below in tables and diagrams. In the second series of test photographs, two glass plates and three films were treated similarly and the results will be shown below in concentrated form. It should be noted that the films in the two series were of different manufacture and that one of the tasks of the tests was to compare the geometrical qualities of the image coordinates in the glass plate and in the films.

(2) Adjustment of Affine Deformations of Image Coordinates. As was mentioned above, film negatives sometimes become affine deformed due to different shrinkage in orthogonal directions. This affine deformation is a systematic error which can be determined in connection with the adjustment procedure. In the basic correction equations (1 and 2 above), the terms for dc are substituted by $\frac{x}{c}dc_x$ and $\frac{y}{c}dc_y$ in the expressions for x and y, respectively. This procedure has been treated in detail in publication (1) and will, therefore, not be repeated here. Only the expressions for the dc_x, dc_y, dr_x, and dr_y will be given in addition to the expression for the determination of the standard error of unit weight.

We find:

$$dc_x = -\frac{c\sqrt{2}}{4r} N_{51x} \tag{16}$$

$$dc_y = -\frac{c\sqrt{2}}{4r} N_{51y} \tag{17}$$

$$dr_x = \frac{\sqrt{2}}{4} N_{51x} \tag{18}$$

$$dr_y = \frac{\sqrt{2}}{4} N_{51y} \tag{19}$$

$$[vv] = [dxdx] + [dydy] - \frac{[dx]^2 + [dy]^2}{5} - \frac{N_{51x}^2 + N_{51y}^2}{4} - \frac{N_{52}^2}{8} -$$

$$- 0.21 \left\{(0.8\ [dx] - N_{53})^2 + (0.8\ [dy] - N_{54})^2\right\} \tag{20}$$

Table III. Grid Tests of Central Perspectives, 5 Points

H-t

302 •   •402

301 •   y   •401

x

101 •   •201

102 •   •202

Circle: 6 (Points 5, 106, 206, 306, 406)
Radius r = 128 mm
Camera const.(appr.)c= mm

$\frac{r^2+c^2}{3r^2} = \frac{39488}{49152} = 0.803$

$\frac{2r^2+5c^2}{12r^2} = \frac{148288}{196608} = 0.754$

dr = 0.177N51 = -0.001 mm

$\frac{\sqrt{2}}{8}\frac{c}{r} = 0.177\,\frac{c}{r} = 0.210$

$\frac{\sqrt{2}}{8r} = 0.177\,\frac{1}{r} = 0.001164$

$\frac{c}{12r^2} = 0.0833\,\frac{c}{r^2} = 0.000773$

$dx = x_{meas.} - x_{given}$ $dy = y_{meas.} - y_{given}$

Point	dx	N51		N52		N53		Ntr		S	
		k	kdx	k	kdx	k	kdx	k	kdx	k	kdx
5	0										
1	-3	-1	+3	+1	-3	+1	-3	+1	-3	+3	-9
2	-14	+1	-14	+1	-14	+1	-14	-1	+14	+3	-42
3	+11	-1	-11	-1	-11	+1	+11	-1	-11	-1	-11
4	+2	+1	+2	-1	-2	+1	+2	+1	+2	+3	+6
[dx] =	-4	N51x =	-20	N52x =	-30	N53x =	-4	N54x =	+2	Sx	-56
[dx²] =	330										
	dy										
5	0										
1	+4	-1	-4	-1	-4	+1	+4	+1	+4	+1	+4
2	-7	-1	+7	+1	-7	-1	+7	+1	-7	+1	-7
3	+10	+1	+10	-1	-10	-1	-10	+1	+10	+1	+10
4	+1	+1	+1	+1	+1	+1	+1	+1	+1	+5	+5
[dy] =	+8	N51y =	+14	N52y =	-20	N53y =	+2	N54y =	+8	Sy	+12
[dy²] =	166	-N51x	-20	+N52x	-30	+N53x	-4	+N54x	+2		+12
		N51	-6	N52	-50	N53	-2	N54	+10		

Corrections:

$$dx_o = -\frac{r^2+c^2}{3r^2}[dx] + \frac{2r^2+5c^2}{12r^2}N_{53} = 0.803 \times 4 - 0.754 \times 2 = +1.7 \text{ microns}$$

$$dy_o = -\ \text{"}\ [dy] + \text{"}\ N_{54} = -0.803 \times 8 + 0.754 \times 10 = +1.1 \text{ microns}$$

$$dc = -\frac{\sqrt{2}}{8}\frac{c}{r}N51 = 0.210 \times 6 = +1.3 \text{ microns}$$

$$d\kappa = -\frac{\sqrt{2}}{8r}N52 = 0.001164 \times \frac{50}{1000} = +0.000058 \text{ rad.} = 0^g0037$$

$$= 0^g0003$$

$$d\varphi = \frac{c}{12r^2}(5N53 - 4[dx]) = 0.000773\,\frac{(-10+16)}{1000} = \frac{0.000773 \times 6}{1000} = 0.000005 \text{ rad.} =$$

$$= -0^g0008$$

$$d\omega = \text{"}\ (4[dy] - 5N54) = \frac{0.000773}{1000}(32-50) = -0.000773 \times 0.018 = -0.000014 \text{ rad.} =$$

$$[vv] = [dx^2] + [dy^2] - \frac{[dx]^2 + [dy]^2}{5} - \frac{N51^2 + N52^2}{8} - \frac{(5N53-4[dx])^2 + (5N54-4[dy])^2}{120} =$$

$$= 496 - \frac{80}{5} - \frac{2536}{8} - \frac{360}{120} = 496 - 16 - 317 - 3 = 160$$

$s_o = \frac{1}{2}\sqrt{[vv]} = 0.0063$ mm   Conf.limits (5 per c.): $0.6s_o - 2.9s_o = 0.004 - 0.018$mm

$$s_o = \sqrt{\frac{[vv]}{10-7}} = \sqrt{\frac{[vv]}{3}} \tag{21}$$

$$s_{s_o} = 0.41 s_o \tag{22}$$

From Table III, we find the following data:

$$dr_x = 7 \text{ microns}$$

$$dr_y = -4.9 \text{ microns}$$

$$[vv] = 496 - 16 - 149 - 312 - 4 = 15$$

$$s_o = \sqrt{5} = 2.2 \text{ microns}$$

Consequently, in the chosen circle (6) of the actual film, there is a certain affine deformation. The standard error of unit weight is considerably reduced through the affine adjustment. In this manner, all measurements in the films were treated and, mainly for comparison, also the measurements in the glass plates. Next, the results obtained from the measurements and discrepancies of Table II will be shown (see Table IV).

Table IV. Radial Distortion and Transformed Radial Distortion; Standard Errors of Unit Weight Before and After Affine Adjustment

Radius (mm)	Prel. Rad. Dist. (micr)	Corrections (microns)	Final Radial Dist. (micr)	Standard Error of Unit Weight (micr)	After Affine Adjustment (micr)
20	9.2	-1.4	7.8	2.5	1.6
41	10.1	-2.9	7.2	3.3	3.2
64	14.5	-4.5	10.0	5.4	1.4
88	6.2	-6.2	0.0	5.2	1.3
106	0	-7.5	-7.5	6.6	5.6
128	-1.1	-9.0	-10.1	6.3	2.2

The transformation of the preliminary radial distortion amounts was made to the 88-mm radius in order to facilitate the comparison with the radial distortion curves from the glass plates and the other films, which all were referred to the 88-mm radius as zero radius for the distortion. The transformation was made proportional to the radii.

(3) Adjustment of Discrepancies in Nine Points. In order to increase the reliability in the determination of the standard error of unit weight, it is desirable to use more redundant observations. For this purpose, nine points seem to be suitable since a considerable increase of the number of redundant observations is obtained but yet the work is relatively limited. If one of the points is located in the neighborhood of the principal point and eight points are symmetrically located on a circle around the center point, the solution of the normal equations becomes rather simple and can be made in a general algebraical way.

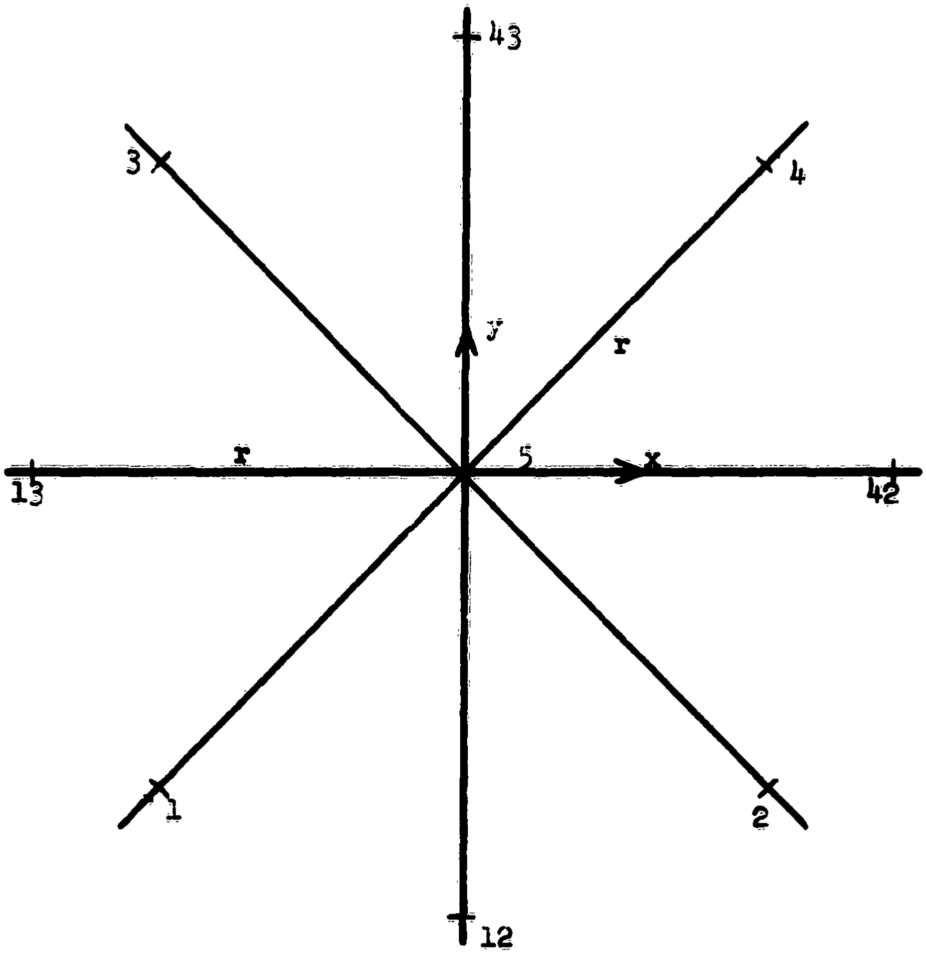


Fig. 2. Locations and notations of the nine points for adjustment. Different radii r can be used. Note that point 13 is located between 1 and 3, 42 between 4 and 2, etc.

For the locations of the points as indicated in Fig. 2, the following corrections of the elements of orientation are obtained from the coordinate errors dx and dy, defined as measured minus given values.

$$dx_o = -\frac{r^2 + c^2}{5r^2}[dx] + \frac{4r^2 + 9c^2}{40r^2} N92$$

$$dy_o = -\frac{r^2 + c^2}{5r^2}[dy] + \frac{4r^2 + 9c^2}{40r^2} N94$$

$$dc = -\frac{c\sqrt{2}}{16r} N91$$

$$d\varkappa = -\frac{\sqrt{2}}{16r} N92$$

$$d\varphi = \frac{c}{40r^2}(9N93 - 8\,[dx])$$

$$d\omega = \frac{c}{40r^2}(8\,[dy] - 9N94)$$

The radial distortion

$$dr = \frac{\sqrt{2}}{16} N91$$

$$[vv] = [dxdx] + [dydy] - \frac{[dx]^2 + [dy]^2}{9} - \frac{N91^2 + N92^2}{16} -$$

$$- \frac{(9N93 - 8[dx])^2 + (9N94 - 8[dy])^2}{720}$$

The standard error of unit weight

$$s_o = \sqrt{\frac{[vv]}{18-6}} = \sqrt{\frac{[vv]}{12}}$$

The standard error of the standard error of unit weight

$$s_{s_o} = \frac{s_o}{\sqrt{24}} = 0.20 s_o$$

The confidence limits are

for the 5-percent confidence level: $0.7s_o - 1.7s_o$

for the 1-percent confidence level: $0.65s_o - 2.0s_o$

The symbols are defined as follows:

$$dx = dx_1 + dx_2 + dx_3 + dx_4 + dx_5 + dx_{12} + dx_{13} + dx_{42} + dx_{43}$$

$$dy = dy_1 + dy_2 + dy_3 + dy_4 + dy_5 + dy_{12} + dy_{13} + dy_{42} + dy_{43}$$

$$N91 = - dx_1 + dx_2 - \sqrt{2}\, dx_{13} + \sqrt{2}\, dx_{42} - dx_3 + dx_4 - \sqrt{2}\, dy_{12} - dy_1 - dy_2 + dy_3 + dy_4 + \sqrt{2}\, dy_{43}$$

$$N92 = \sqrt{2}\, dx_{12} + dx_1 + dx_2 - dx_3 - dx_4 - \sqrt{2}\, dx_{43} - dy_1 + dy_2 - \sqrt{2}\, dy_{13} + \sqrt{2}\, dy_{42} - dy_3 + dy_4$$

$$N93 = dx_1 + dx_2 + 2dx_{13} + 2dx_{42} + dx_3 + dx_4 + dy_1 - dy_2 - dy_3 + dy_4$$

$$N94 = dx_1 - dx_2 - dx_3 + dx_4 + 2dy_{12} + dy_1 + dy_2 + dy_3 + dy_4 + 2dy_{43}$$

The weight and correlation numbers are:

$$Q_{x_o x_o} = Q_{y_o y_o} = \frac{4r^4 + 9c^4 + 4r^2c^2}{20r^4}$$

$$Q_{cc} = \frac{c^2}{8r^2}$$

$$Q_{\varkappa\varkappa} = \frac{1}{8r^2}$$

$$Q_{\varphi\varphi} = Q_{\omega\omega} = \frac{9c^2}{20r^4}$$

$$Q_{x_o\varphi} = - Q_{y_o\omega} = \frac{c}{20r^4}(4r^4 + 9c^2)$$

$$Q_{rr} = \frac{1}{8}$$

A form for the computation of the symbols and the corrections has been worked out. In Table V, an example of the computations in the form is shown. The figures have been taken from the Table II. As indicated above, the nine-point combination is primarily to be used for a more reliable determination of the standard error of unit weight and for the determination of residuals in other points. For the routine work in connection with calibration procedures, the five-point combination is the most suitable one.

It would also be possible to treat all five-point combinations and the nine-point combination simultaneously. All elements of orientation except the dc should be identical for all solutions and a normal equation system should be set up, containing the five parameters dx_o, dy_o, $d\kappa$, $d\varphi$, and $d\omega$ and in addition the dc_1 dc_2... etc. for the individual circles. If affine solutions are to be made, the dc_1 etc. would be substituted by dc_{x1} and dc_{y1} etc.

c. Results of Multicollimator Tests of Two Different Cameras, Using Glass Plates and Two Different Makes of Film. The cameras are called A and B. Each of them was calibrated in the U. S. Geological Survey multicollimator. From each camera, one glass plate and one strip of film were received. The image coordinate measurements were made in the Mann Comparator Nr 62 10 05 at GIMRADA, Fort Belvoir. The computations were made according to the principles treated above. The results of the measurements and computations are shown below in tables and diagrams.

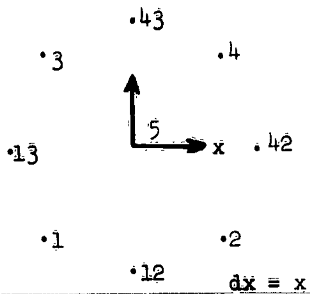
(1) Negatives from Camera A.

(a) Tests of Glass Plate and Films. The obtained radial distortion curve from the glass plate negative is shown in Fig. 3. Also, the two affine radial distortion curves were determined and are shown together with the average curve. There are minor differences between the curves, of the order of magnitude of one to two microns. Even if a certain systematic tendency can be found, the differences are too small for a significant affine deformation. Local shrinkages of the emulsion and minor creepings from the drying procedure might be the explanation.

In Fig. 3 also, the standard errors of unit weight are shown for the different radii and also before and after the affine adjustment. There are minor improvements of the accuracy from the affine adjustment but they are very limited.

In Fig. 4, the radial distortion curves from the glass plate and from the two film negatives are shown for comparison. The agreement is very good. The maximum deviations between the results from the plate and from the films is of the order of magnitude of 2 microns. The corresponding principal distances as computed from

Table V. Grid Tests of Central Perspectives 9 Points



Circle: 4
Radius r = 88 mm
Camera const. c = 152 mm

$$\frac{r^2 + c^2}{5r^2} = 0.7967$$

$$\frac{4r^2 + 9c^2}{40r^2} = 0.7713$$

dr = 0.088N91 = 95 microns

$$\frac{\sqrt{2}c}{16r} = 0.088\frac{c}{r} = 0.1520$$

$$\frac{\sqrt{2}}{16r} = \frac{0.088}{r} = 0.00100$$

$$\frac{c}{40r^2} = 0.000491$$

dx = x meas. - x given; dy = y meas. - y given

Point	dx microns	N91 k	N91 kdx	N92 k	N92 kdx	N93 k	N93 kdx	N94 k	N94 kdx	S k	S kdx
5	0										
1	-7	-1	+7	+1	-7	+1	-7	+1	-7	+3	-21
2	-7	+1	-7	+1	-7	+1	-7	-1	+7	+3	-21
3	+1	-1	-1	-1	-4	+1	+1	-1	-1	-1	-1
4	+4	+1	+4	-1	-4	+1	+4	+1	+4	+3	+12
12	-6		+1.4	-8.4						+2.4	-14.4
13	-8	-1.4	+11.2			+2	-16			+1.6	-12.8
42	+6	+1.4	+8.4			+2	+12			+4.4	+26.4
43	+4			-1.4	-5.6					-0.4	-1.6
[dx]	-13	N91x=	+22.6	N92x=	-33.0	N93x=	-13	N94x=	+3	Sx=	-33.4
[dx²]	267										-33.4
	dy										
5	0										
1	-3	-1	+3	-1	+3	+1	-3	+1	-3	+1	-3
2	-8	-1	+8	+1	-8	-1	+8	+1	-8	+1	-8
3	+15	+1	+15	-1	-15	-1	-15	+1	+15	+1	+15
4	+6	+1	+6	+1	+6	+1	+6	+1	+6	+5	+30
12	-14	-1.4	+19.6					+2	-28	+1.6	-22.4
13	+9			-1.4	-12.6					-0.4	-3.6
42	-3			+1.4	-4.2					+2.4	-7.2
[dy]	+26	N91y=	+85.2	N92y=	-30.8	N93y=	-4	N94y=	+30	Sy=	+106.4
[dy²]	1196	N91x N91	+22.6 +107.8	N92x N92	-33.0 -63.8	N93x N93	-13 -17	N94x N94	+3 +33		+106.4

Corrections:

$$dx_o = -\frac{r^2 + c^2}{5r^2}[dx] + \frac{4r^2 + 9c^2}{40r^2} N93 = -2.8 \text{ micr.}$$

$$dy_o = - \text{ '' } [dy] + \text{ '' } N94 = +4.7 \text{ micr.}$$

$$dc = -\frac{c\sqrt{2}}{16r} N91 = -16.4 \text{ microns}$$

$$d\varkappa = -\frac{\sqrt{2}}{16r} N92 = 6.1^{cc}$$

$$d\varphi = \frac{c}{40r^2}(9N93 - 8[dx]) = -15^{cc}$$

$$d\omega = \text{ '' } (8[dy] - 9N94) = -28^{cc}$$

$$[vv] = [dx^2] + [dy^2] - \frac{[dx]^2 + [dy]^2}{9} - \frac{N91^2 + N92^2}{16} - \frac{(9N93 - 8[dx])^2 + (9N94 - 8[dy]^2)}{720} = 374$$

$$S_o = \sqrt{\frac{[vv]}{12}} = 5.6 \text{ microns}$$

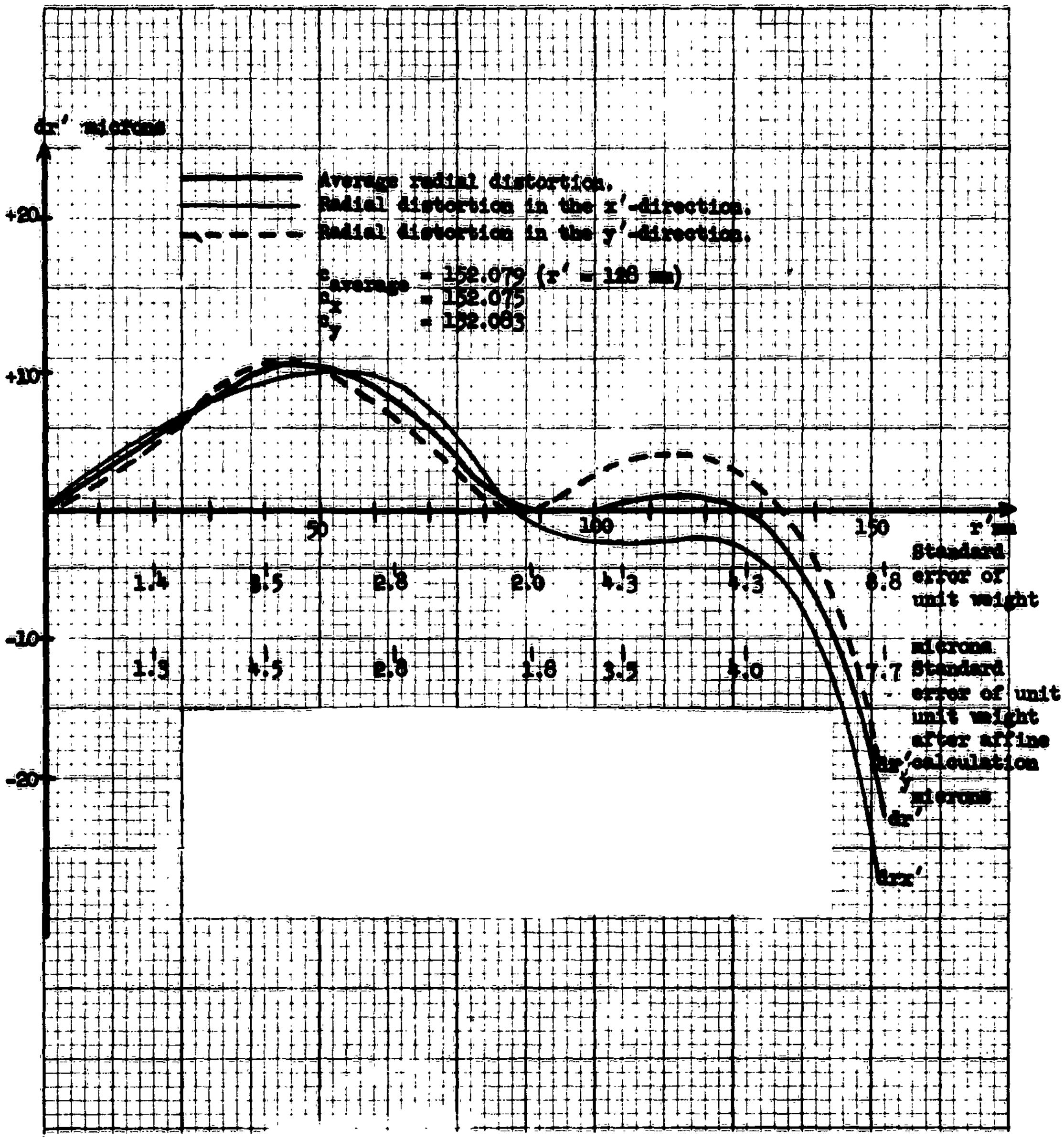


Fig. 3. Results of tests of image coordinates from multicollimator photographs. Camera A. Glass plate negative. Radial distortion curves and standard errors of unit weight for varying radii.

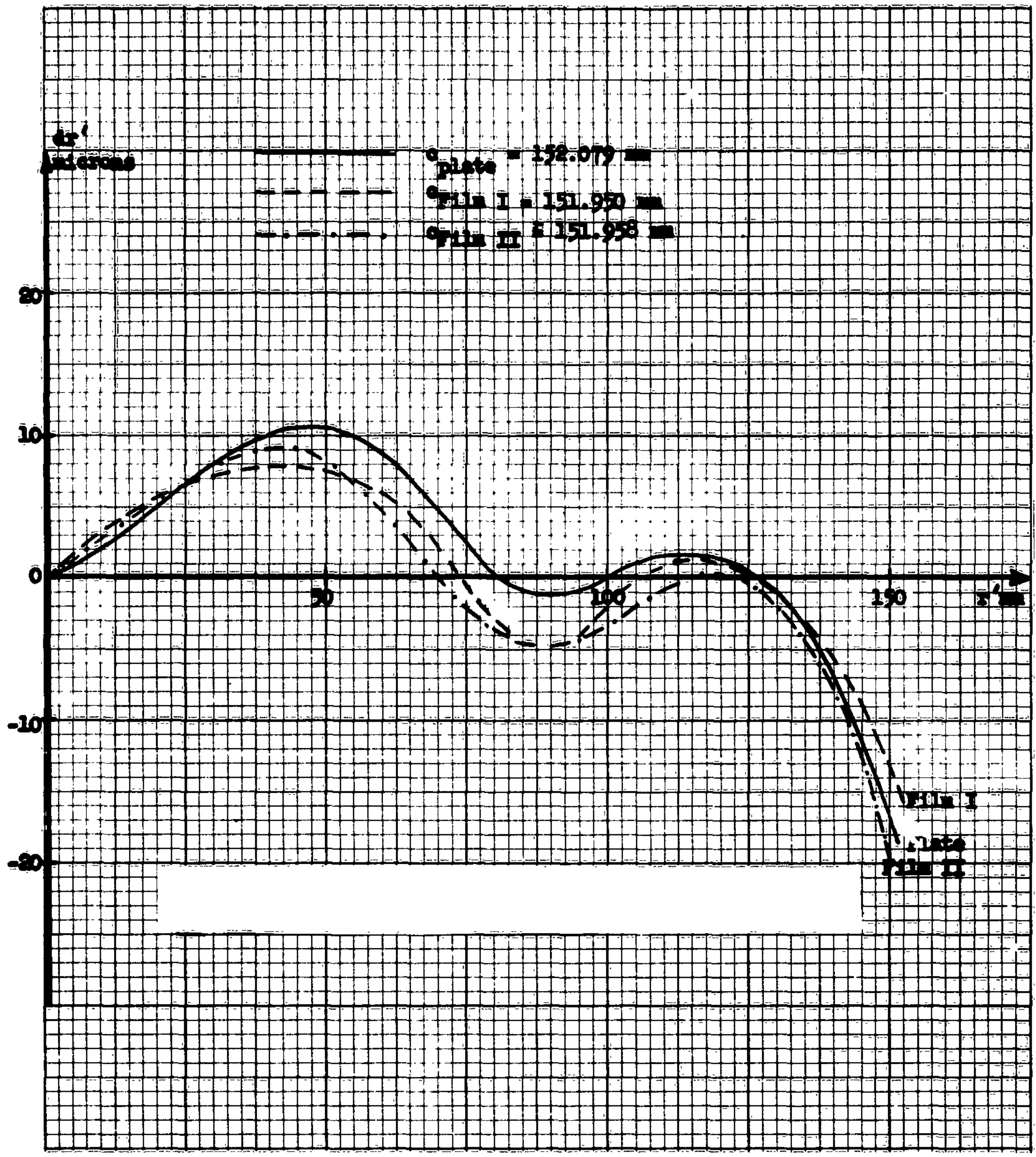


Fig. 4. Radial distortion curves from glass plate and two Films I and II. Camera A.

identical circles show considerable shrinkages of the films in comparison with the glass plate, however. The standard errors of unit weight of the image coordinates from the films proved to be very large and increased with the radii. Adjustment including corrections for the affine deformations proved that very pronounced affine shrinkages were present in the film negatives. The results of these computations are shown in Fig. 5. The two film samples agree very well and show that the affine shrinkage probably is consistent throughout the strip. From the standard errors of unit weight after the affine adjustments, weights were computed for the different radii as inverse proportional to the squares of the standard errors. The weight in the principal point was chosen as unity and was determined after extrapolation of the standard errors of unit weight. The weight distribution as a function of the radii is shown in Fig. 6. For the glass plate as well as for the two films, there is a clear decrease of the weights with increasing radii. It is of great interest to find that this weight variation, which has been found in tests of aerial photographs, is present also in laboratory tests. The lacking flatness of the negative surface is probably the main cause of the weight variation. It is of great interest to find that the geometrical quality of the glass plate negative and the two film negatives after affine adjustment is nearly identical. This confirms earlier experience from high tower tests, where the accuracy of glass negatives and films proved to be approximately the same after affine adjustment of the film negative discrepancies.

Finally, a summary is made of the corrections to the elements of orientation of the camera for the glass plate and film tests (see Table VI). In the summary, the standard errors of unit weight of the image coordinates before and after the affine adjustment are also shown. The accuracy of the determination of the elements of orientation varies considerably with the radius of the actual circle combination. In order to show the theoretical standard errors to be expected in the determination of the elements of orientation, the following figures are computed for the largest circle (radius = 152 mm) and for the standard error of unit weight 0.01 mm. The expressions in equation (15) were used.

$$Q_{x_o x_o} = Q_{y_o y_o} = \frac{11}{6} = 1.8333 \qquad s_{x_o} = s_{y_o} = 0.013 \text{ mm}$$

$$Q_{\varkappa\varkappa} = \frac{1}{4r^2} \qquad s_{\varkappa} = \frac{s_o}{2r} = 0.000033 \text{ rad.} = 0^{g}0021$$

$$Q_{\varphi\varphi} = Q_{\omega\omega} = \frac{5c^2}{6r^4} \qquad s_{\varphi} = s_{\omega} = 0.000060 \text{ rad.} = 0^{g}0038$$

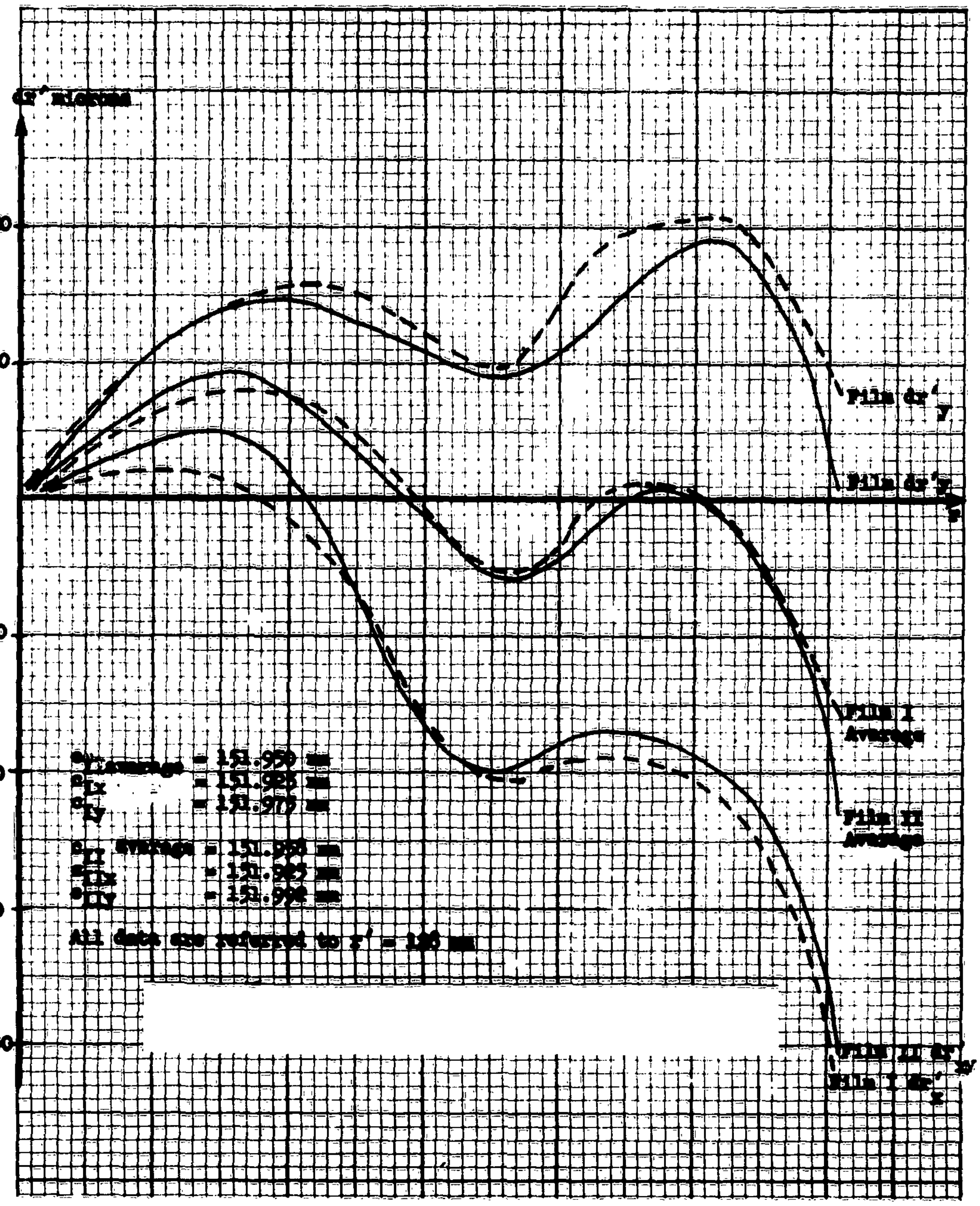


Fig. 5. Affine radial distortion curves from Films I and II and the averages. Camera A.

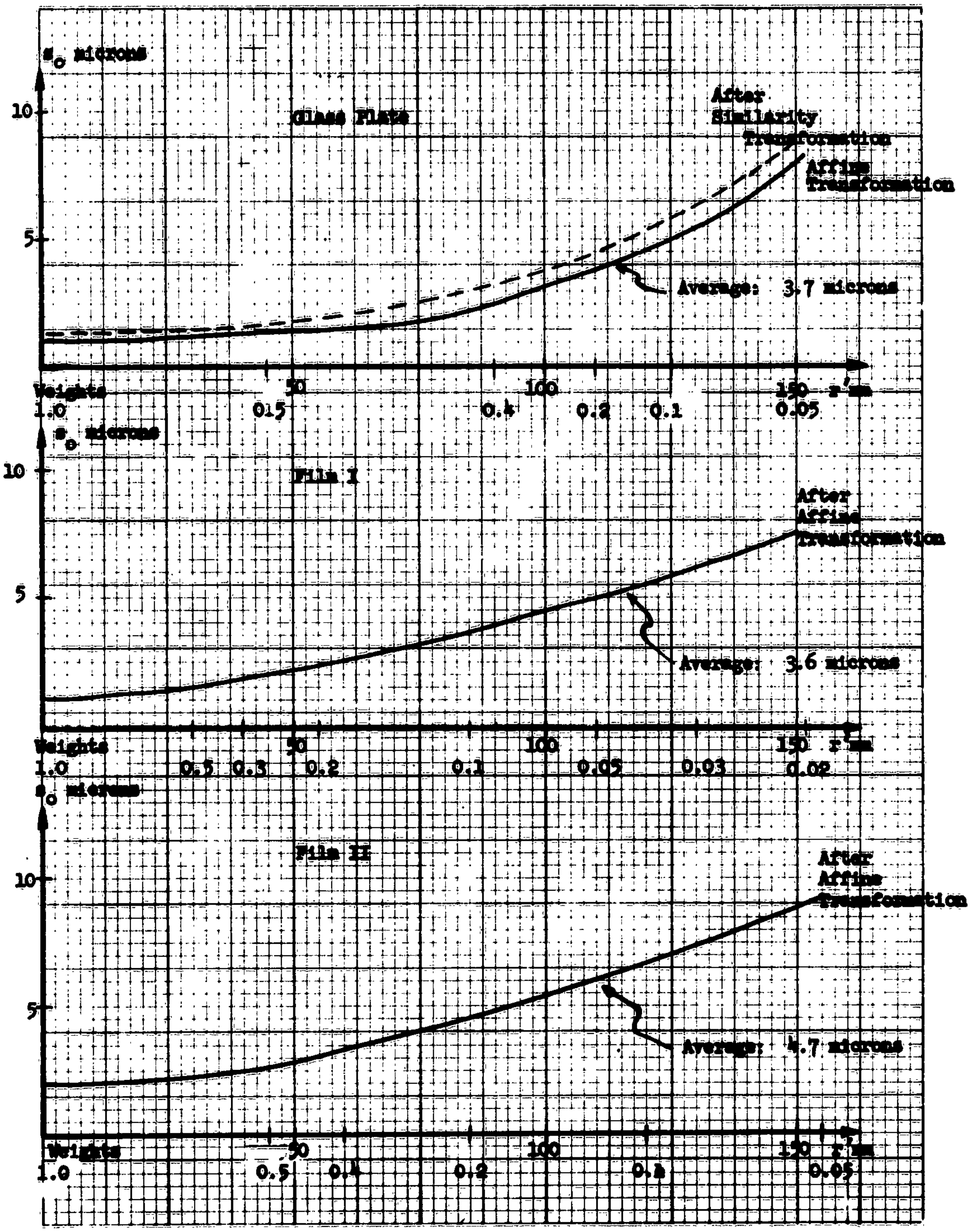


Fig. 6. Weight variations of the image coordinates.

From Table VI, the great variation of the corrections from the small circles is evident. When the circles become larger, the values stabilize clearly. There is a pronounced difference between the corrections dφ, dω from the plate and from the films, respectively. It seems possible that the camera orientation has become somewhat changed when the film magazine was inserted.

Table VI. Corrections of Elements of Orientation of Camera from Treatment of Glass Plate P and Film Negatives FI and FII

Neg.	Radius (mm)	dx_o (microns)	dy_o	dκ (radians 10^{-6})	dφ (radians 10^{-6})	dω	s_o (micr.)	s_o After Affine Adjustment (microns)
P	20	+196	0	-9	+32	-3	1.4	1.2
FI		+24	-168	+71	+126	-1138	3.7	1.4
FII		-80	-48	-247	-505	-252	3.0	2.4
P	41	+45	+23	-234	-121	-286	3.5	4.5
FI		+3	+10	+91	+23	-68	6.6	0.9
FII		-26	+1	-267	-158	-23	5.1	2.5
P	64	0	-3	-3	+15	+3	2.8	2.8
FI		+29	-38	+78	+197	+59	10.4	3.4
FII		+4	-44	-248	+59	+257	11.0	6.0
P	88	+6	+6	-148	+57	-51	2.0	1.8
FI		+2	-2	+41	+131	+151	15.0	2.9
FII		+6	-1	-274	+57	+70	14.5	2.5
P	106	0	+6	-160	+18	-47	4.3	3.5
FI		+20	-33	+100	+135	+212	18.6	3.5
FII		+15	-21	-292	+115	+135	14.6	2.4
P	128	+4	-4	-155	+29	+1	4.3	4.0
FI		+3	-37	+73	+33	+274	19.3	7.0
FII		+6	-20	-286	+53	+131	19.2	3.7
P	152	-1	-7	-144	+8	+26	8.8	7.7
FI		0	-54	+39	+38	+318	27.5	6.2
FII		+6	-31	-279	+53	+201	23.0	10.1

(b) <u>Comparison Between Image Coordinate Measurements in Two Film Samples</u>. In order to see the statistical distribution of the differences between two independently measured film samples, the two sets of measurements were transformed into one and

the same system via the measurements in the center point 5 and the four fiducial marks. After this transformation, which was made including parameters for affinity and lacking orthogonality between the coordinate axes, the residual coordinate errors were computed for all points in the two coordinate directions. The statistical distribution of these errors or differences was then tested with the usual normal distribution test. (See the Appendix.) The results are shown in Fig. 7. The differences are evidently very well normally distributed and can consequently be regarded to be of irregular (accidental or random) character.

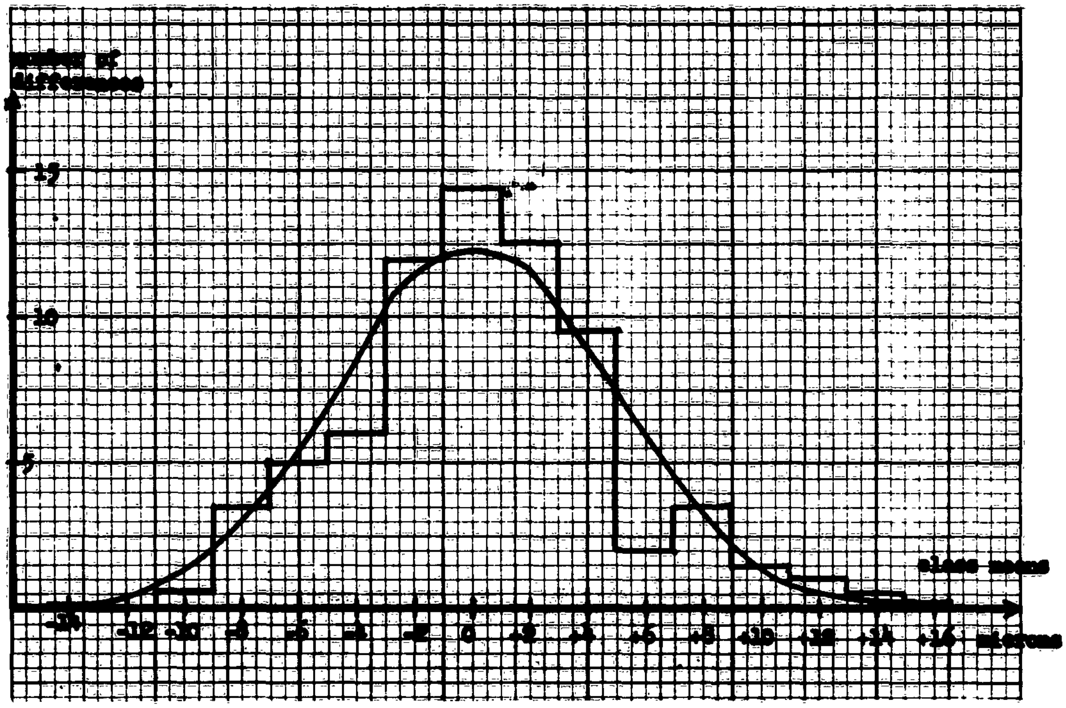


Fig. 7. Histogram and normal distribution curve of coordinate differences between two films from a multicollimator test and after a coordinate transformation with the aid of the fiducial marks. The differences are very well normally distributed.

(2) Negatives from Camera B. The results of the radial distortion determination from the glass plate negative are shown in Fig. 8. A determination of affine radial distortion curves was also made as shown in the figure. The differences between the curves are small, however, and no significant affine deformation was found. The curves are somewhat irregular in comparison with what could be expected from the actual type of lens. The standard errors of unit weight will be presented together with corresponding data from the films. In this case, three film negatives were selected from the strip. Two negatives were chosen in the ends of the strip

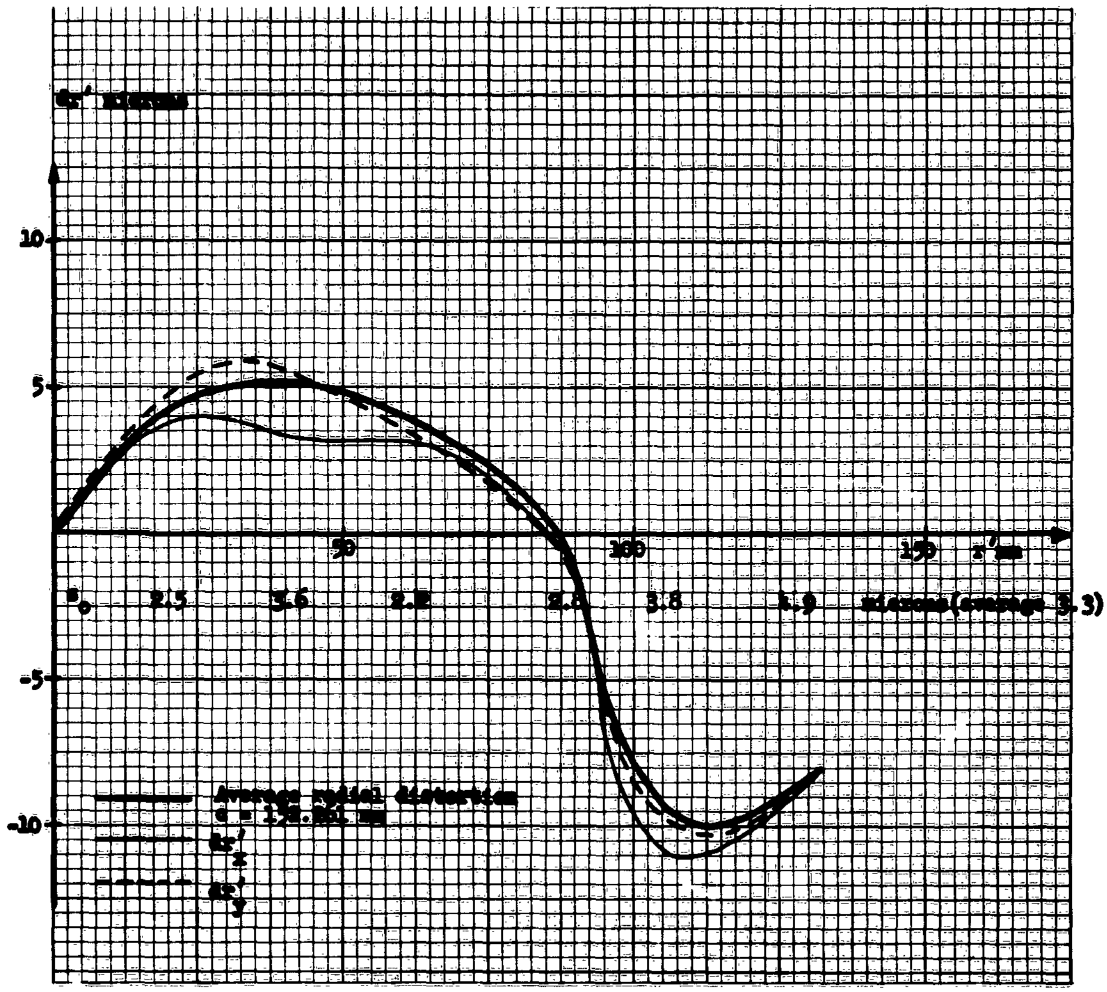


Fig. 8. Multicollimator tests. Glass plate negative. Radial distortion curves and standard errors of unit weight for different radii.

and the third one in the middle. In Figs. 9 through 11, the results of the radial distortion determinations from the film negatives are shown. There is a certain affine deformation in all three negatives but it is very small in comparison with the films from the camera A. Further, in Fig. 12, the radial distortion curves from all four negatives are shown for comparison. There is a considerable difference between the curve from the glass plate negative and from the films. Since the differences are so great that they may be significant, a special investigation will be made according to ordinary statistical procedures.

The maximum difference is about 4 microns. Next, the standard errors of the radial distortion amount are computed. The weight number of the radial distortion is $\frac{1}{4}$ according to (15). The

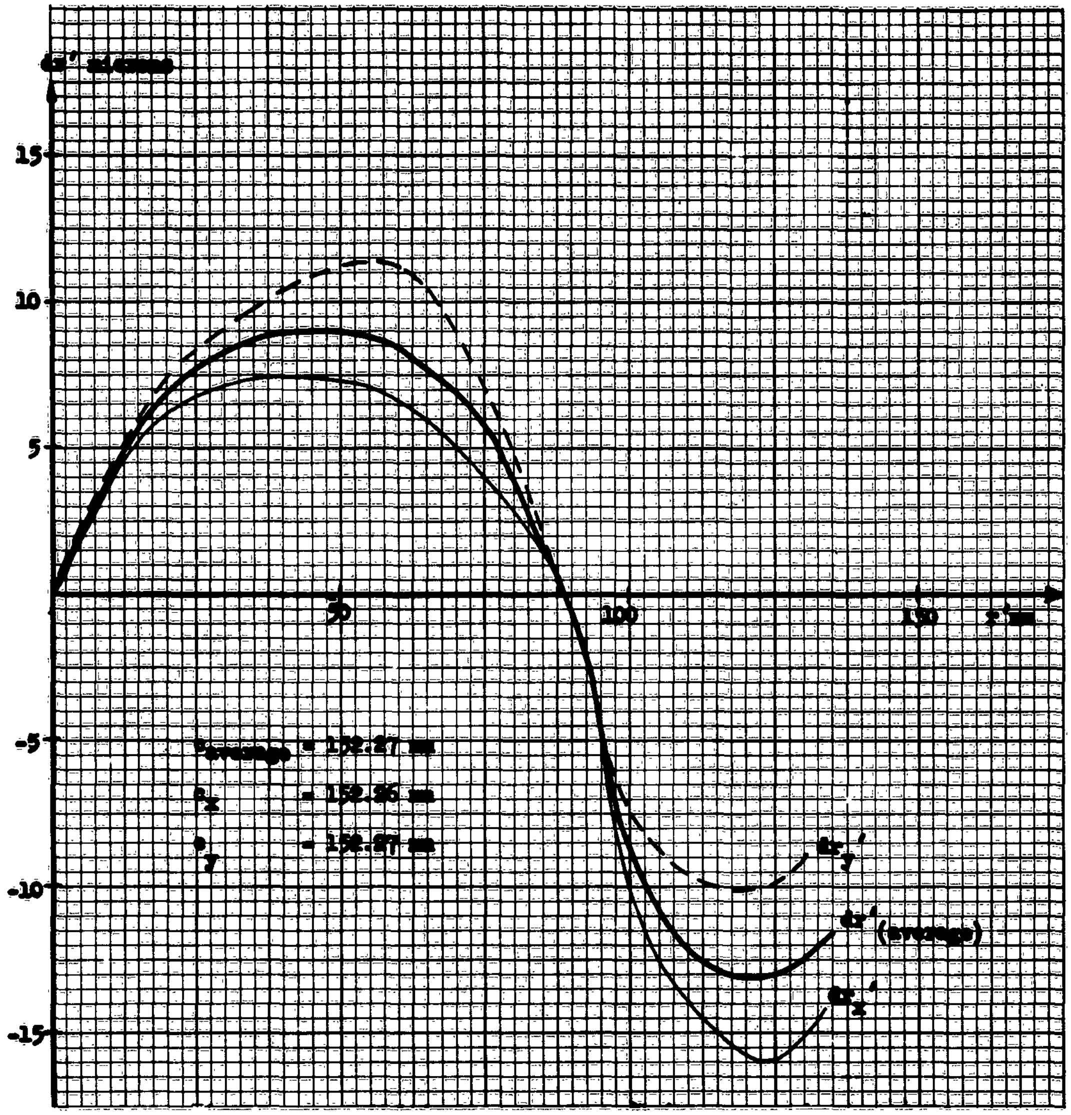


Fig. 9. Multicollimator tests (Cronar film). Camera B. Affine radial distortion curves. Film I (end of the strip).

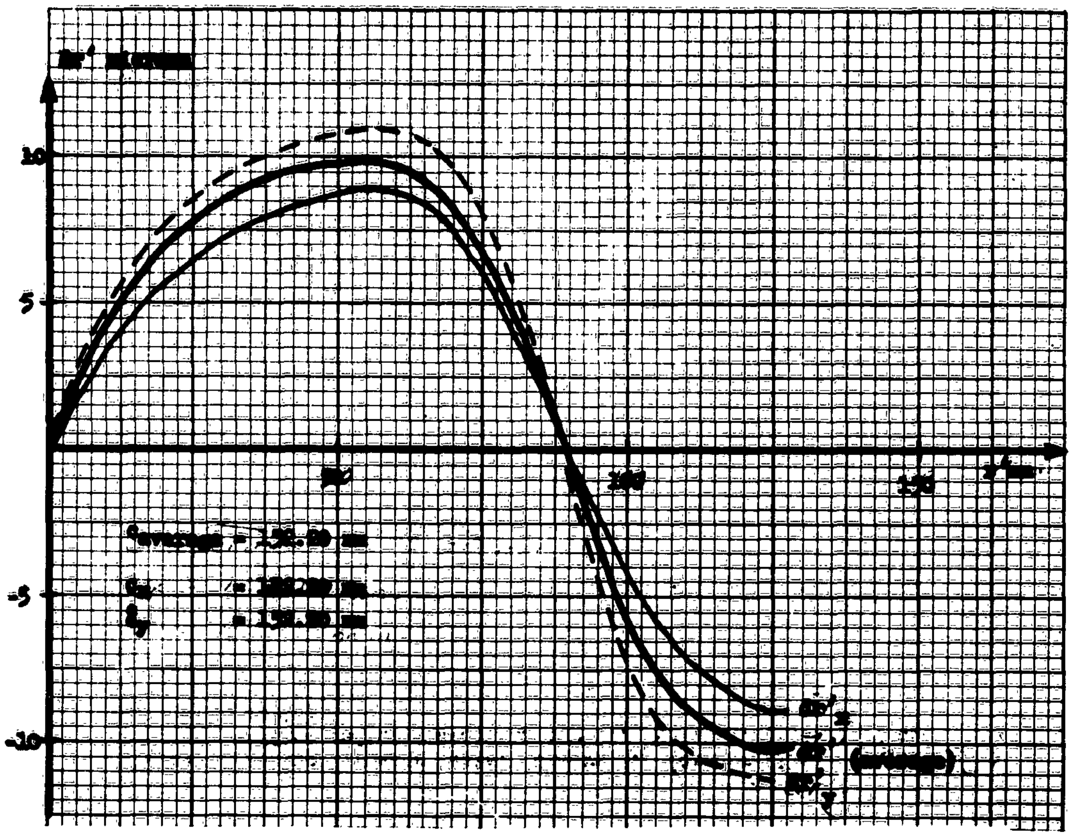


Fig. 10. Multicollimator Tests. Camera B (Cronar film). Affine radial distortion curves. Film II (middle of the strip).

standard errors of unit weight are 3.3 and 4.6 microns for the actual circles in the glass plate and film, respectively. The standard errors of the distortion amount from the glass plate and the film are, consequently, 1.6 and 2.3 microns, respectively. For the test, the following expression is next computed (see reference 9):

$$s_d^2 = \frac{s_1^2(N_1 - 6) + s_2^2(N_2 - 6)}{N_1 + N_2 - 12}$$

For $N_1 = N_2 = 10$, $s_1 = 1.6$ and $s_2 = 2.3$ microns, we find $s_d = 1.95$ microns. Further,

$$t = \frac{d}{s_d} \sqrt{\frac{N_1 N_2}{N_1 + N_2}} = 4.7$$

From a table of the t-distribution, we find the level to be between 0.1 and 0.2 percent. This means that the difference 4 microns under actual circumstances is to be regarded as significant. There are

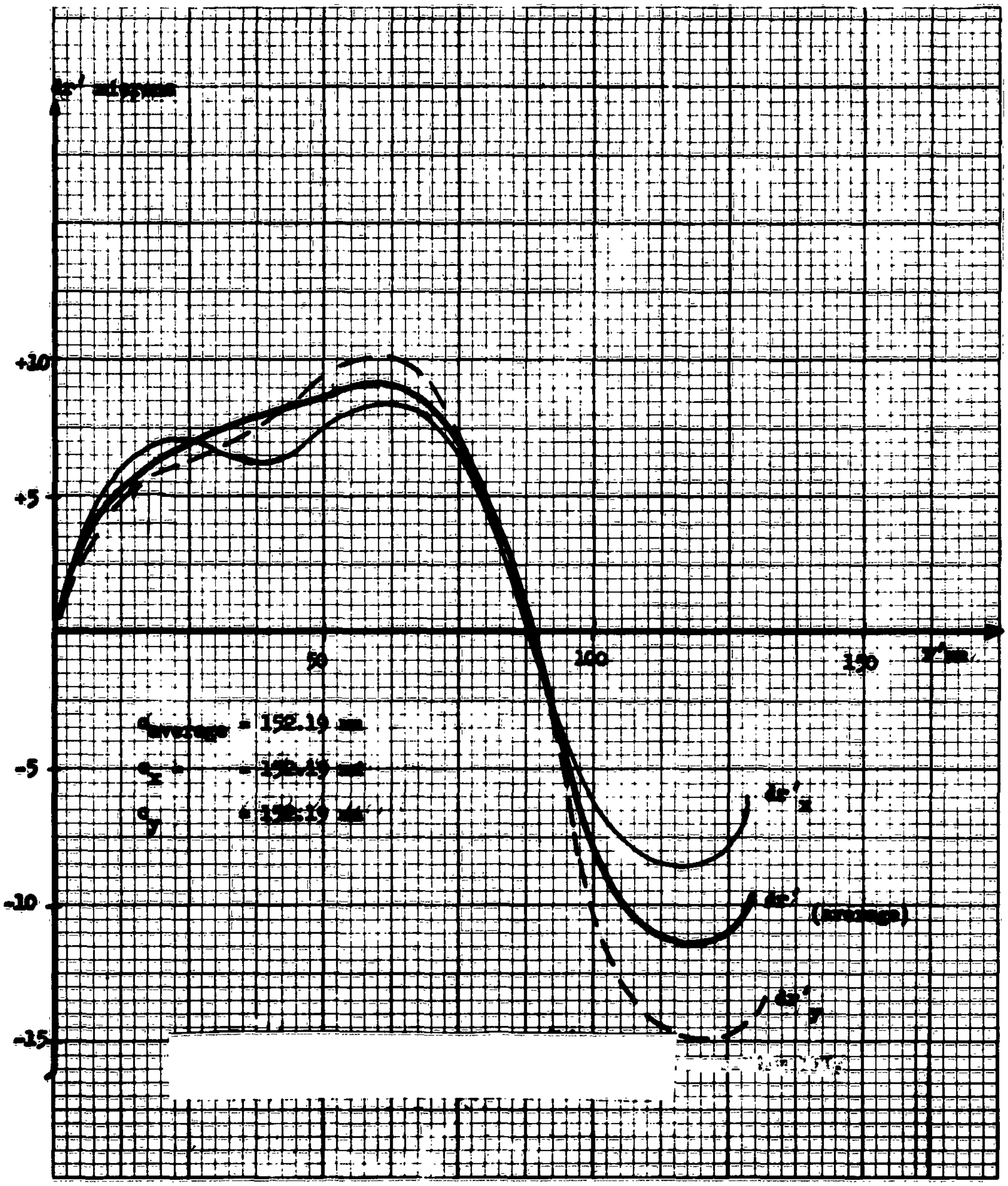


Fig. 11. Multicollimator tests (Cronar film). Affine radial distortion curves. Film III (end of strip).

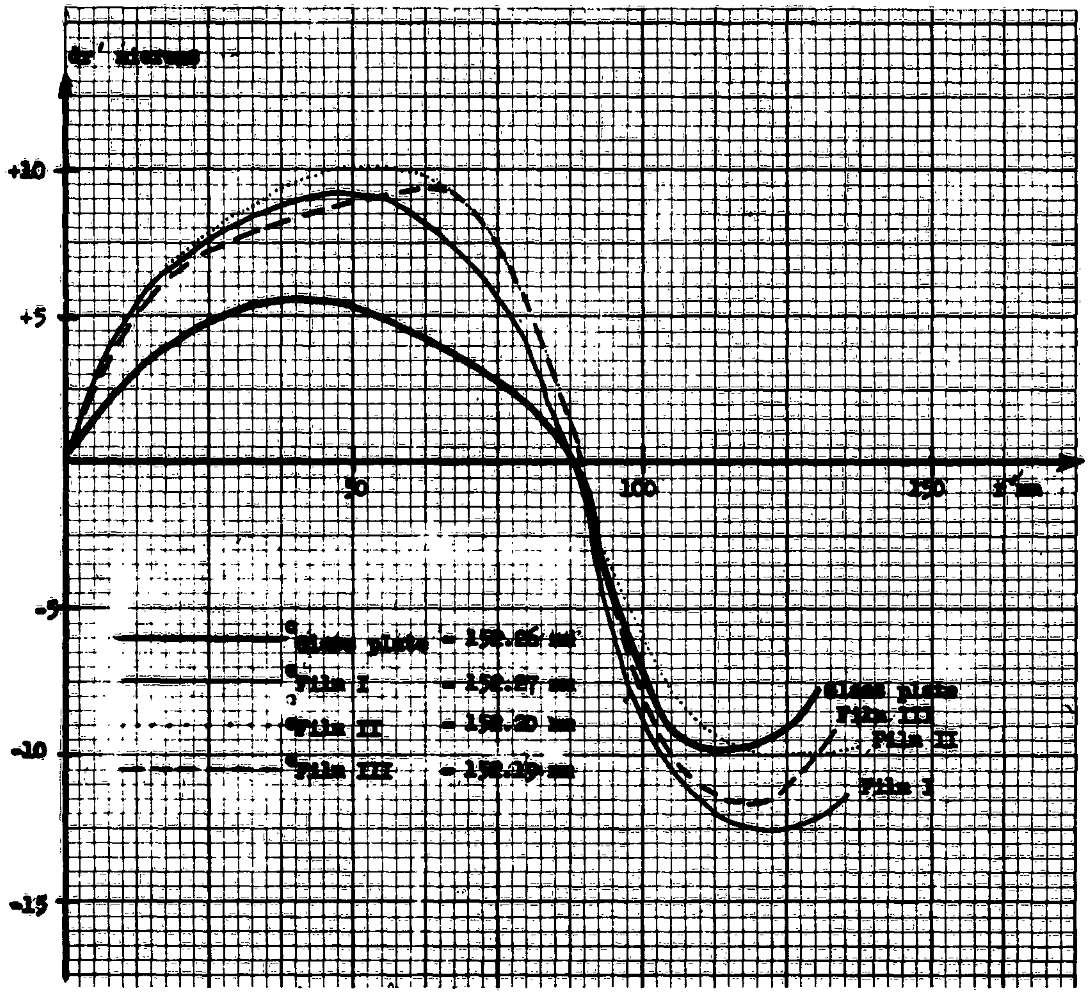


Fig. 12. Multicollimator tests. Radial distortion curves from glass plate and film negatives (Cronar) Camera B.

some approximations in the procedure used but, for the purpose of deciding the significance, the reliability of the test ought to be satisfactory.

There may be several reasons for the deviation in the distortion glass plate curve from the film distortion curves. The most probable causes are deviations from flatness either of the glass plate or of the supporting back of the camera when film was used. Since the films agree so well mutually, local deformations of the film are less probable.*

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* A test of an additional, specially made glass plate negative is presented below.

Next, comparisons between the standard errors of unit weight from the glass plate and the film negatives will be made. In Table VII, the standard errors of unit weight before the affine adjustments are summarized. The corresponding weight distribution of the image coordinates is shown in Fig. 13. From Table VII, it is evident that the film samples from camera B are of very good quality in comparison with the samples from camera A as shown in Table VI. The film quality is not much lower than the quality of the glass plate.

Table VII. Standard Errors of Unit Weight from Films and Glass Plate before Affine Adjustment (All data given in microns)

Radius (mm)	Film I	Film II	Film III	Average	Plate
20	1.3	2.5	2.1	2.0	2.5
41	2.6	3.3	4.4	3.4	3.6
64	3.6	5.4	5.5	4.8	2.2
88	7.2	5.2	6.3	6.2	2.8
106	3.1	6.6	5.3	5.0	3.8
128	4.8	6.3	5.2	5.4	4.9
			Averages	4.6	3.3

Table VIII. Standard Errors of Unit Weight from Films after Affine Adjustment (All data given in microns)

Radius (mm)	Film I	Film II	Film III	Average
20	1.5	1.6	2.2	1.8
41	2.1	3.2	2.4	2.6
64	2.2	1.4	2.0	1.9
88	8.4	1.3	1.8	3.8
106	2.2	5.6	3.4	3.7
128	7.8	2.2	2.9	4.3
			Average	3.0

In general, a certain decrease of the standard errors of unit weight is found after the affine adjustment (see Table VIII). In some individual cases, however, a pronounced increase was found, in particular concerning film I. This may be explained by the comparatively low reliability of the determination of the standard error of unit weight. There are only three redundant observations (degrees of

freedom) in the adjustment procedure and, therefore, the reliability becomes rather low. This is also expressed by the concept standard error of the standard error of unit weight which in this case becomes about 41 percent of the standard error itself (see expression 22 above). More detailed information about the importance of the degrees of freedom can be obtained from a calculation of the confidence limits according to the t- and chi^2-distributions. The weight variations of the image coordinates after affine adjustment are shown in Fig. 13.

It is evident from Tables VII and VIII that the basic geometrical quality of the films after affine adjustment is practically the same as that of the glass plate. In both cases, the standard error of unit weight of the image coordinates is of the order of magnitude 3 microns.

Table IX summarizes the corrections of the elements of the exterior orientation from the glass plate and the films and for the different circles. It is evident that the corrections become more stabilized for the circles with the larger radii. This is in good agreement with the expressions for the weight numbers (15) where the

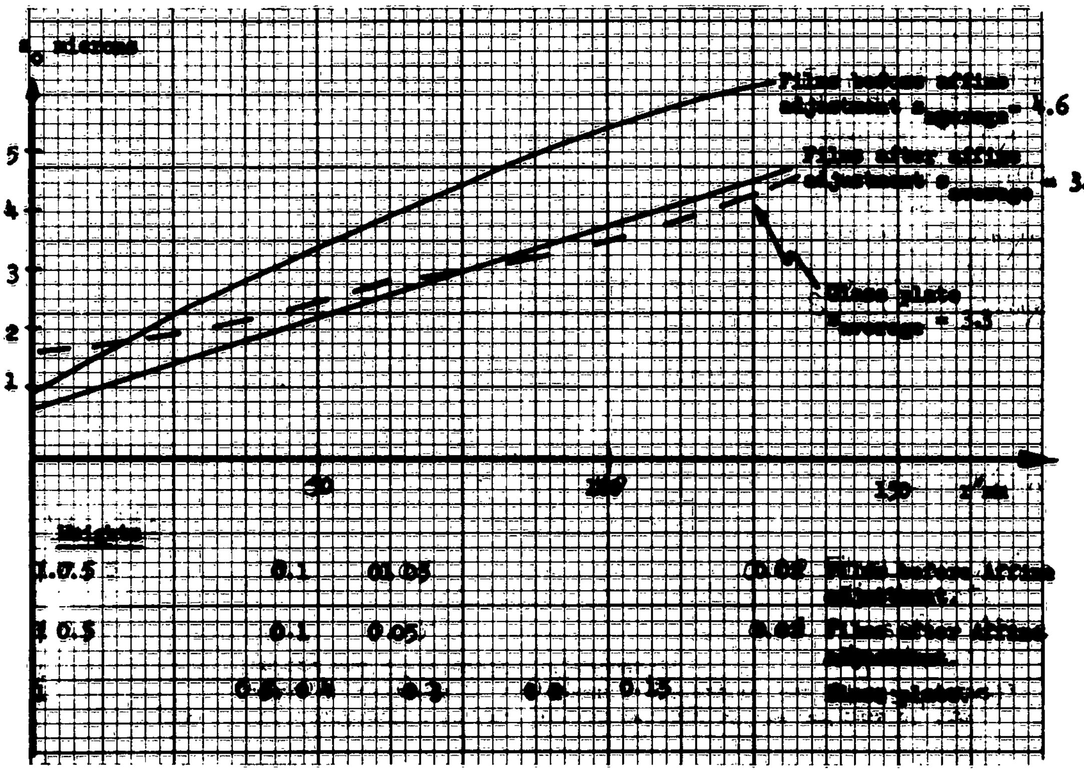

Fig. 13. Multicollimator tests. Investigations of the weights of the image coordinates of the glass plate and of the films (Cronar), before and after affine adjustments.

Table IX. Corrections of Elements of Exterior Orientation

Negative	Circle Radius (mm)	dx_o (micr)	dy_o (micr)	$d\kappa$ (cc)	$d\varphi$ (cc)	$d\omega$ (cc)
Glass plate	20	-68	-36	+101	-322	+200
	41	+26	+5	+86	+91	-15
	64	+16	-1	+53	-98	+36
	88	-9	+13	+68	-51	-59
	106	-11	+10	+74	-58	-62
	128	-8	+9	+67	-43	-43
Film I	20	+72	-124	+39	+300	+495
	41	+51	+51	+49	+210	-210
	64	+25	-17	+37	+104	+72
	88	+7	+11	+18	+15	-37
	106	-20	-3	+26	-71	-1
	128	-8	-4	+18	-29	+12
Film II	20	+80	-496	+78	+325	+400
	41	+1	+47	+47	+5	+196
	64	-14	+15	+39	-67	+59
	88	-6	+5	+42	-30	-26
	106	-36	+1	+51	-33	+18
	128	+2	+2	+44	+3	+9
Film III	20	+92	-52	-45	+480	+320
	41	+5	+30	-49	-28	+105
	64	+28	+4	-25	-116	-16
	88	-63	+15	-18	-51	-60
	106	-38	+16	-29	-16	-12
	128	+1	0	-21	-11	-4

radius appears in the denominator. The weight numbers become consequently smaller for increasing radii, and the standard errors decrease with the square root of the weight numbers. For comparison, the standard errors of the actual elements of orientation are shown for the largest circle r = 128 mm and for the standard error of unit weight 0.005 mm.

$$s_{x_o} = s_{y_o} = 0.006 \text{ mm}$$

$$s_\kappa = 12^{cc} \qquad \text{(centesimal seconds)}$$

$$s_\varphi = s_\omega = 27^{cc} \qquad \text{"} \qquad \text{"}$$

For the smallest circle (r = 20 mm) and the standard error of unit weight 0.002 mm, we find

$$s_{x_o} = s_{y_o} = 0.108 \text{ mm}$$

$$s_{\kappa} = 32^{cc}$$

$$s_{\varphi} = s_{\omega} = 444^{cc}$$

(a) <u>Additional Test of a Glass Plate Negative</u>. In order to investigate closer the significant difference in the radial distortion curve from the glass plate and the film negatives, another glass plate negative was made at the U. S. Geological Survey and was put at disposal. The test procedure was identical with those applied and described above. The results of the test are shown in Fig. 14 and in Table X. From the figure, it is evident

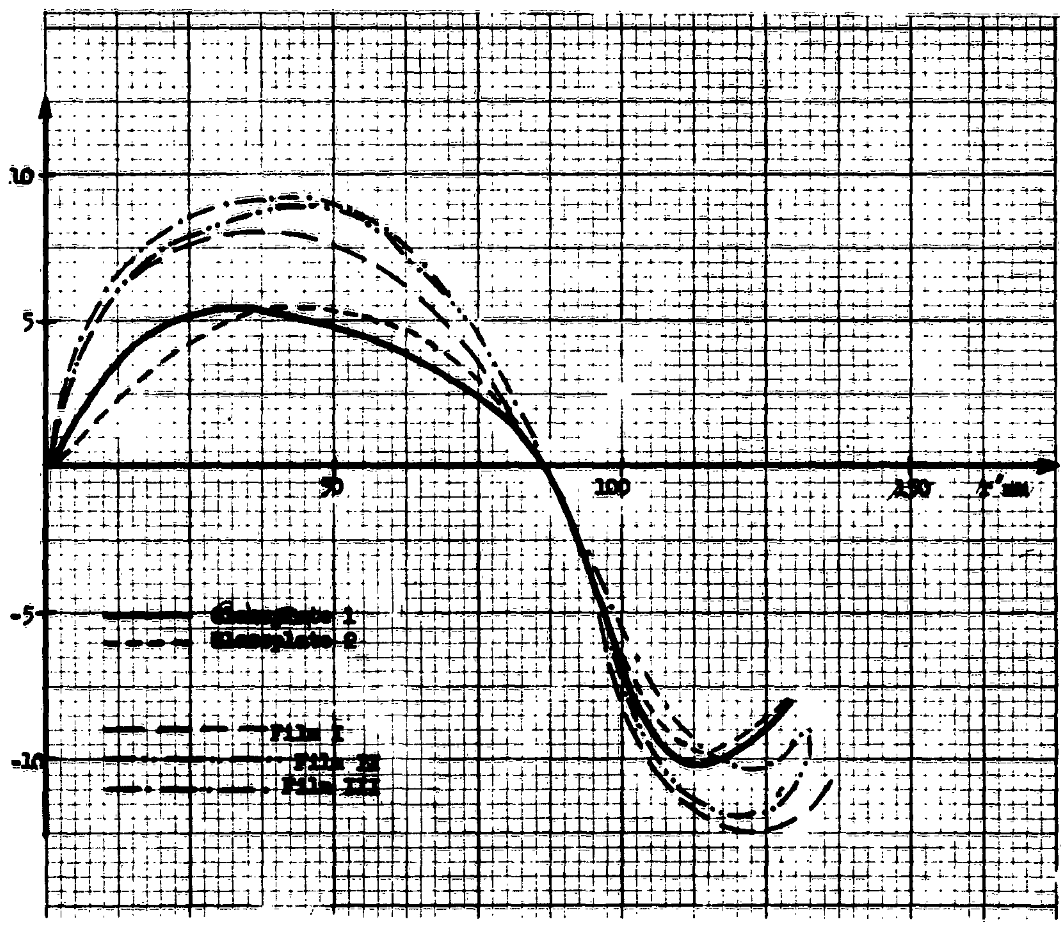

Fig. 14. Multicollimator tests. Aviogon 118. c = 152 mm. Radial distortion curves from glass plate and film negatives. There is a significant difference which may be due to a curved supporting back.

that the two glass plates agree very well mutually and that the difference between the distortion curves as found above is confirmed. Since it is hardly possible that two glass plates would have identical deformations, the most probable reason for the differences is a curved supporting back of the film magazine. The standard errors of unit weight within the six circles as shown in Table X are of the same order of magnitude as from the test of the first glass plate negative (see Table VII).

Table X. Standard Errors of Unit Weight from Adjustment of Measurements in Second Glass Plate

Radius (mm)	s_0 (microns)
20	1.8
41	7.0
64	1.1
88	4.4
106	2.0
128	4.8
Average	3.5

(b) Determination of Principal Distance of Photographs with Some Different Methods. The nominal camera constant is 152.24 mm for the zero-radius 75 mm according to the manufacturer of the camera. From the measurements of the coordinates of the fiducial marks and computations of the distances between these marks and comparison with the nominal distances, the following values (Table XI) of the image constants are obtained from the glass plate and the films.

Table XI. Determinations of Image Constants

Direction	Glass Plate (mm)	Film I (mm)	Film II (mm)	Film III (mm)	Average Films (mm)
Length	152.25	152.24	152.18	152.17	152.19
Across	152.24	152.25	152.18	152.17	152.20
Average	152.24	152.24	152.18	152.17	152.20

The films show a minor overall shrinkage as an average. No information concerning the standard error of the nominal camera constant is available from the manufacturer. From the multicollimator test and

the adjustment, the following values of the principal distance were found for the circle r = 88 mm as zero-radius.

Glass plate	152.26 mm	
Film I	152.27 mm	
Film II	152.20 mm	Average 152.22 mm
Film III	152.19 mm	

The standard error of the determination can be obtained from the standard error of unit weight and the weight number according to the expression

$$s_c = s_o \frac{c}{2r}$$

(see (15) above). From Table VII, we find the standard errors of unit weight to be 6.2 and 2.8 microns for the radius 88 mm and for the films and the glass plate, respectively. The standard errors of c become: for the glass plate: $s_c = 0.003$ mm; and for the film: $s_c = 0.005$ mm. For the radius 75 mm, we finally find: $c_{glass} = 152.25$ mm; and $c_{film} = 152.21$ mm. The agreement with the manufacturer's value of c is rather good.

(3) Comparison between Geometrical Quality and Photographic Resolving Power of Tested Negatives. Each of the targets for the geometrical calibration in the multicollimator is combined with a target for a test of the photographic resolving power. It is a full-contrast line-pattern of the usual type where the thickness and the density of the lines is gradually changed. The resolution is defined in lines per mm according to the last figure where the lines still can be distinguished. Such a test was applied to the films and plates, which were used for the determination of the geometrical accuracy in order to find possible a relation between this quality and the photographic resolution. The averages of the values for the results from camera A are shown in Table XII.

Table XII. Comparison between Geometrical Accuracy and Resolving Power (Camera A)

Circle	Radius (mm)	Resolving Power (lines/mm)	Standard Error of Unit Weight (microns) Average
1	20	24	1.7
2	41	24	2.6
3	64	20	4.1
4	88	20	2.4
5	106	20	3.1
6	128	20	4.9
7	152	17	8.0

In Table XIII, the corresponding values are shown for camera B.

Table XIII. Comparison between Geometrical Accuracy and Resolving Power (Camera B)

Circle	Radius (mm)	Resolving Power (lines/mm)	Standard Error of Unit Weight (microns) Average
1	20	44	2.2
2	41	48	4.8
3	64	42	1.7
4	88	38	3.6
5	106	35	2.9
6	128	27	4.8

In the results from both the cameras, there is a clear correlation between the resolving power and the standard error of unit weight. The resolving power decreases and the standard error of unit weight increases with the distance from the center of the image. In other words, the decrease of the weight of image coordinates with the radius, which has been discovered earlier and which has been confirmed in these investigations, is closely related to the decrease of the resolving power. Therefore, it seems probable that the decrease of the weight to a certain extent may be caused by the decreasing resolving power but there are doubtless many more causes of the weight variation. The most important one might be the lacking flatness of the image surface which always will have the mentioned effect when the observations and measurements of the image coordinates are made orthogonally.

It is evident from the tables above that there is a considerable difference in the resolving power between cameras A and B. The averages of the standard errors of unit weight of the two cameras are also different.

(4) Summary of Multicollimator Tests of Negatives from Aerial Cameras. The applied method for the determination of the geometrical quality of the negatives in connection with the camera calibration procedure in the multicollimator has proved to be very convenient and to give valuable results. Again, the method of least squares has shown one of its most important properties: to distinguish between regular and irregular errors in discrepancies between given and measured data. In particular, a well-defined procedure for the determination of the elements of the interior orientation has been established. The geometrical quality of the elements of the interior orientation can be defined and expressed theoretically

correctly as standard errors in terms of the basic standard errors of unit weight of the observations and the corresponding weight and correlation numbers. This is of great importance for all functions of the elements of the interior orientation, i.e. practically all results of photogrammetric measurements. It is, however, most important to remember that the standard error of unit weight of the image coordinates as determined and shown above refer to laboratory conditions and that the corresponding data from operational conditions certainly are different. Above, the standard error of unit weight of the image coordinates was found to be of the order of magnitude of 3 microns for each of the two image coordinates and as an average for the entire photograph. A certain variation of this standard error was found for different parts of the photographs indicating a weight variation of considerable amount and very similar to what has been found through tests of aerial photographs from operational conditions.

From such tests, the basic standard error of unit weight has been found to be of the order of magnitude of 6 to 8 microns in film negatives and as an average for the entire photograph. A very pronounced variation of the standard error of unit weight with the radius has been found and, consequently, a considerable variation of the weights of the image coordinates (see publication 2). This weight variation has been confirmed from the calibration tests above and has, consequently, been found to be present already in the calibration stage of the aerial camera. Some possible reasons for the weight variation will be treated in more detail below.

So far, it is important to use 6 to 8 microns as a realistic value of the standard error of unit weight of the image coordinates in aerial wide-angle photographs instead of the obtained value 3 to 4 microns from the calibration tests. The tests of the aerial photographs, however, are rather limited up to now and have been performed exclusively with film negatives. Such tests should be performed much more frequently and should be applied to different types of cameras and to film and glass plate negatives. The procedure for the tests is very similar to what has been applied to the collimator tests. It should also be noted that the same procedure can be applied to tests of all kinds of central perspectives, including photographs for non-topographic applications, microphotographs, X-ray photographs, and television images.

Concerning the detailed results from the performed multicollimator tests, the obtained accuracy must be regarded to be high. The standard errors of unit weight contain error influences from the adjustment of the multicollimator itself, from the imaging procedure, from the measurements in the comparator, and from the operator. In this case, the operator was not very well trained. The different qualities of the two tested film types could be well distinguished

and proved a very pronounced affine shrinkage in one of the films. The standard error of unit weight after correction for the affine deformation became of nearly the same order of magnitude for both the films and does not differ considerably from the corresponding standard error of unit weight of the glass plate negative coordinates. This is a remarkable fact, indicating that the basic accuracy of films and glass plates in these experiments has proved to be nearly the same. It would be most interesting to investigate the corresponding relations after aerial test photography and also under different conditions concerning the photographic treatment of the films and glass plates. It seems very possible that, in particular, the drying process for films and glass plates may have considerable influence upon the geometrical qualities of the image coordinates. It would also be of great value to make similar tests using different focussing settings of the multicollimator for the determination of a possible correlation between the resolving power of the optical system and the photographs on one hand and the basic geometrical accuracy of the image coordinates on the other. It seems that such an empirical procedure is the most effective method for treating this very important but difficult problem, at least in the introductory stage. Later, when the general relations between resolving power and geometrical precision and accuracy are experimentally tested, more detailed studies of the relation between individual causes of the changes in the resolving power and the geometrical qualities can be made in a similar way.

It would also be very interesting and valuable to use the derived methods for investigations of the relation between different makes of glass plates and films where the graininess, thickness of emulsions, spectral sensitivity, exposure time, and development are varied. In this way, certainly valuable information on the relation between the basic geometrical qualities and the mentioned factors would be obtained. Next, some applications of the derived method for the test of the interior orientation are treated.

4. Determination of Standard Errors of Elements of Interior Orientation. The elements of the interior orientation are primarily the principal point and the principal distance. Further, the disturbances of the central projection must be known for the purpose of the reconstruction of the bundle of rays on the object side of the lens. Such disturbances are generally identified with regular errors of the image coordinates such as the radial distortion, the tangential distortion, affine deformations, and other possible coordinate errors which follow a certain well-defined law. Finally, the irregular errors of the image coordinates, defined and expressed as the standard error of unit weight after a well-defined calibration procedure should also be determined and shown. For all elements which have been determined through the calibration adjustment procedure, i.e. those elements to which corrections have been computed from the

adjustment in terms of the measured quantities (the discrepancies of the image coordinates), the accuracy can be determined and expressed as a standard error. A summary of the standard errors of the corrections from the normal equations was given for the radius 152 mm above in paragraph 3c(1). The figure 0.01 mm was used as standard error of unit weight. It is evident that, due to the weight variation of the image coordinates within the photographic image, the standard error of unit weight increases with the radius but the weight numbers decrease with increasing radius. Therefore, there must be a certain radius which is the most favorable to use for the determination of the correction of the actual element. There is, in other words, a problem to determine that radius which gives the smallest standard error of the element in question. For an application of this principle to the calibration in practice, the weight variation within the photograph has to be determined more completely than up to now and more tests of the same type as shown above are necessary. In order to show how the principle would work in practice, we will use a preliminary expression for the weight variation. In publication (2), the standard error of unit weight of the image coordinates was found to follow this equation:

$$s_o = 1 + 0.008r + 0.00028r^2$$

where s_o is the standard error of unit weight in microns for the radius r in millimeters.

The weight number of the correction dc of the principal distance is as shown in (15)

$$Q_{cc} = \frac{c^2}{4r^2}$$

The standard error of r can, consequently, be written

$$s_r = s_o \frac{c}{2r} \qquad \text{or}$$

$$s_r = (1 + 0.008r + 0.00028r^2) \frac{c}{2r} \qquad \text{or}$$

$$s_r = \frac{c}{2r} + 0.004c + 0.00014rc$$

Next, such a value of r is to be determined which makes s_r a minimum. Applying the usual procedure in such a case, we find

$$\frac{ds_r}{dr} = -\frac{c}{2r^2} + 0.00014c = 0$$

Since the second derivative always is positive, there is a minimum for

$$r = 59 \text{ mm}$$

In other words, for the radius of about 60 mm, the best value of the camera constant would be determined under the mentioned conditions. Similarly, other elements of orientation can be treated. As already said, however, more determinations of the weight variations of the image coordinates are necessary before this principle can be applied to practice. For a determination of the standard error of the coordinates of the principal point, the following procedure is applied. We assume the camera to be adjusted very carefully in the multicollimator and the image of the central collimator target to coincide with the intersection of the two fiducial mark lines or with the point halfway between two of the opposite points. The image of the central collimator target is, consequently, to be regarded as a preliminary position of the principal point. Through the adjustment, corrections will be found to this point in the x- and y-directions. The corrections are to be determined from the differential formulas (1) and (2) where x and y are put = 0. Hence, we find

$$dx = dx_o - cd\varphi$$

$$dy = dy_o + cd\omega$$

The weight numbers of the corrections can be found from the general law of error propagation

$$Q_{xx} = Q_{x_o x_o} + c^2 Q_{\varphi\varphi} - 2cQ_{x_o\varphi}$$

$$Q_{yy} = Q_{y_o y_o} + c^2 Q_{\omega\omega} + 2cQ_{y_o\omega}$$

After substitution of the corresponding expressions from (15), we find after some simple computations

$$Q_{xx} = Q_{yy} = \frac{1}{3}$$

To this weight number, the influence of the original measurements in the preliminary principal point in connection with the coinciding operation has to be added. The standard error of this operation is assumed to be of the same magnitude as the standard error of unit weight of image coordinate measurements and, consequently, the number 1 is added to the weight numbers of the corrections Q_{xx} and Q_{yy}. Hence, the weight numbers of the corrected position of the principal point become

$$Q_{x_p x_p} = Q_{y_p y_p} = \frac{4}{3}$$

and the standard errors of the final coordinates of the principal point become

$$s_{x_p} = s_{y_p} = s_o \frac{2\sqrt{3}}{3}$$

That value which refers to the circle which is used for the determination of the corrections should be used as standard error of unit weight. The standard error of the radial distortion amounts is found from the expression (15)

$$s_{dr} = \frac{s_o}{2}$$

The standard errors of the affine deformations can be found from (16) and (17).

We derive

$$Q_{c_x c_x} = \frac{c^2}{2r^2}$$

$$Q_{c_y c_y} = \frac{c^2}{2r^2}$$

and

$$Q_{dr_x dr_x} = \frac{1}{2}$$

$$Q_{dr_y dr_y} = \frac{1}{2}$$

Hence

$$s_{c_x} = s_{c_y} = s_o \frac{c}{r} \sqrt{\frac{2}{2}}$$

$$s_{dr_x} = s_{dr_y} = s_o \sqrt{\frac{2}{2}}$$

s_{c_x} and s_{c_y} are the standard errors of the camera constants in the x- and y-directions, respectively, as determined from the affine adjustment, and s_{dr_x} and s_{dr_y} are the standard errors of the radial distortion amounts in the two directions.

5. Criterion for Tangential Distortion in Photographic Images. Tangential distortion in photographs was first discovered by Pennington, and a report (10) on this type of regular error in image coordinates was published in 1947. According to this paper, tangential distortion is characterized by the circumstance that a straight line in the object through the camera axis (image of the principal point) becomes imaged as a curve in the photograph. The simplest way to test if tangential distortion is present in a photographic system is, therefore, to image points on straight lines through the camera axis and then to test, through measurements, if the point sequences in the image are sufficiently curved for a statement that tangential distortion is present. Evidently, since such a decision must be founded upon measurements, the geometrical quality of the measurements is of importance. For large amounts of tangential distortion, the basic quality of the measurements may not be very critical since the irregular errors of the measurements may be completely covered by the regular error but the smaller the tangential distortion becomes, the more critical will the distinction be between the two types of errors. Here, some investigations will be made for the decision whether or not tangential distortion can be said to be present in the results of image coordinate measurements of points which are expected to form a straight line through the principal point or, which is the same, if there are sufficient lateral deviations between two sets of point sequences in the object and in the image passing through the principal point.

The test for possible tangential distortion in a photograph can, therefore, first be concentrated to the test of the straightness of a line through coordinate measurements of points along the line. This problem is of importance for photogrammetry in general since straight lines play an important role in several connections. It is sufficient to mention the requirements on straightness of the rods of photogrammetric plotters which must be checked in some way.

a. Determination of Straightness of a Line through Measurements. We assume the points on an assumed straight line to be located at approximately the same distance from a check line, for instance the x-axis of a comparator of high quality (see Fig. 15).

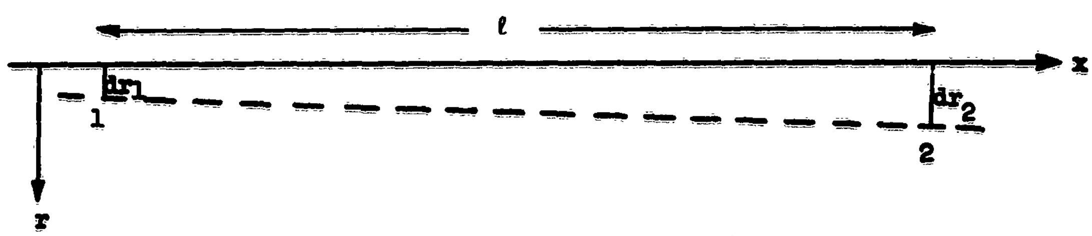

Fig. 15. The coordinates x and r of points along the line ℓ are measured as indicated. The distance between the points 1 and 2 is denoted ℓ.

The coordinates x and r of the points can be measured with replicated settings in each point in order to increase the precision of the averages of the results of the settings. The standard deviation of the averages of the settings in each point are assumed to be equal. Two points 1 and 2 are used as reference points and the straightness of the line ℓ is to be determined with respect to the deviations of the points from the line defined by the two points. It must be remembered that this line is defined by points which are determined by measurements too and that, therefore, the line itself cannot be regarded to be free from errors. These errors have to be taken into account when the question of straightness is to be treated.

In order to make the two points 1 and 2 coincide with the control line, one translation dr_o and one rotation φ are introduced. The influence upon the coordinates r of these two parameters can be expressed by the equation

$$dr = dr_o + xd\varphi \tag{23}$$

From the coordinates dr_1 and dr_2, which are assumed to be of differential order of magnitude and the expression (23), the corrections $-dr_o$, $-d\varphi$ can be obtained according to the two equations

$$dr_1 = - dr_o$$

$$dr_2 = - dr_o - \ell d\varphi$$

where ℓ denotes the length between the points 1 and 2.

From these two expressions, the two corrections are immediately found:

$$dr_o = - dr_1$$

$$d\varphi = \frac{dr_1 - dr_2}{\ell}$$

Next, all the r-coordinates of measured points along the line ℓ are corrected according to (23). The correction to an arbitrary point becomes

$$dr_{corr} = - dr_1 + \frac{x}{\ell}(dr_1 - dr_2) = dr_1\left(\frac{x}{\ell} - 1\right) - dr_2\frac{x}{\ell}$$

If the original r-coordinate in a point to be corrected was dr_x the corrected value or the residual dr_r becomes

$$dr_r = dr_x + dr_1\left(\frac{x}{\ell} - 1\right) - dr_2\frac{x}{\ell}$$

Next, the errors of all measured data are substituted by their statistical values or the standard error of unit weight s_o of the measurements of the coordinates r. The special law of error propagation then gives

$$s_{r_r}^2 = s_o^2 \left\{ 1 + \left(\frac{x}{\ell} - 1\right)^2 + \frac{x^2}{\ell^2} \right\} = 2s_o^2 \left(1 + \frac{x^2}{\ell^2} - \frac{x}{\ell}\right)$$

Hence, the standard error of the residual dr_r after the corrections is

$$s_{r_r} = s_o \sqrt{2\left(\frac{x^2}{\ell^2} - \frac{x}{\ell} + 1\right)} \qquad (24)$$

This expression has a minimum for $x = \frac{\ell}{2}$ i.e. halfway between the two points 1 and 2. The minimum is

$$s_{r_{min}} = 1.22 s_o$$

It is also evident that the standard error increases for a constant x with decreasing ℓ i.e. when the two point 1 and 2 approach each other. Expression (24) is graphically shown in Fig. 16.

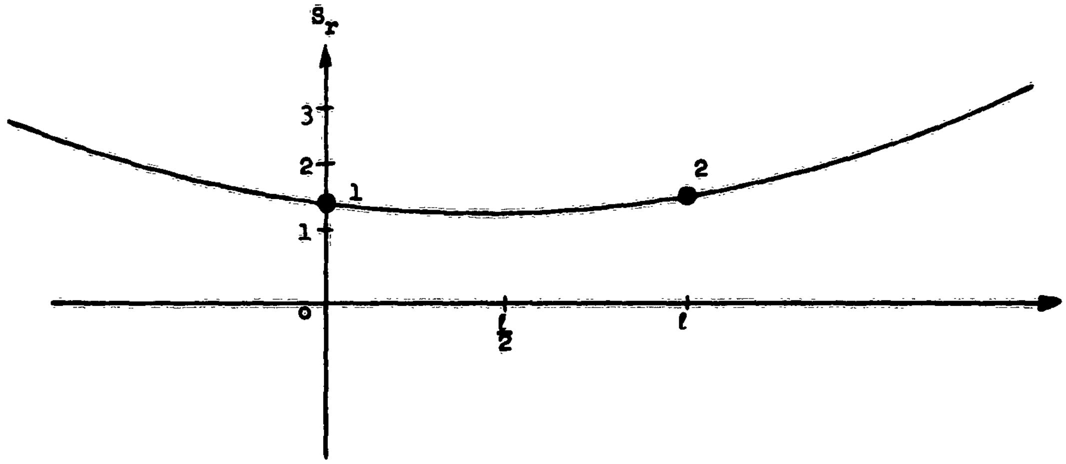

Fig. 16. Variation of the standard error of a corrected deviation. $s_o = 1$.

The root mean square value of the standard errors of the residual deviations within a certain interval can be found with well-known procedures. Between two points x_1 and x_2 we find

$$\text{rmsv} = s_o \sqrt{\frac{1}{x_2 - x_1} \int_{x_2}^{x_1} 2\left(\frac{x^2}{\ell^2} - \frac{x}{\ell} + 1\right) dx}$$

After integration, we find

$$\text{rmsv} = s_o \sqrt{\frac{2}{3\ell^2}(x_2^2 + x_1x_2 + x_1^2) - \frac{x_1 + x_2}{\ell} + 2} \tag{25}$$

For $x_2 = 1$ and $x_1 = 0$ is found

$$\text{rmsv} = s_o \sqrt{\frac{5}{3}}$$

Next, the procedure for the determination of the standard error of unit weight of the measurements is briefly shown.

Measurements of the straightness are assumed to be applied to a line (rule) of very high straightness. The straightness can also be measured in identical points after rotating the rule through 180° around the straight edge. After the two sets of measurements are averaged, the results will represent a line of high straightness. From the inevitable deviations due to the errors of the measurements, the standard error of unit weight is to be determined. For this purpose, the method of least squares is applied, as follows. The differential expression (23) is applied to all measured points. Therefore, it is written in the following manner

$$v = - dr_o - Xd\varphi - dr$$

It should be noted that X refers to the point of gravity of the measured points.

The normal equations are next formed for n points

$$ndr_o + [dr] = 0$$

$$[XX]\, d\varphi + [Xdr] = 0$$

Hence $dr_o = - \dfrac{[dr]}{n}$

$$d\varphi = - \frac{[Xdr]}{[XX]}$$

Further, according to well-known relations

$$[vv] = [drdr] - \frac{[dr]^2}{n} - \frac{[Xdr]^2}{[XX]} \tag{26}$$

The standard error of unit weight is then found from

$$s_o = \sqrt{\frac{[vv]}{n-2}}$$

b. <u>Application to Coordinate Residuals in Test Photographs</u>. We assume that the geometrical quality of a photograph has been determined according to the grid method, for instance in a multicollimator, and that the residual coordinate errors have been computed for the test points. The problem is then to determine if the residuals indicate presence of tangential distortion. It is necessary that residuals can be studied along a line through the principal point, and it is preferable that the line is a diagonal. Since the residual errors usually are given as x- and y-components, they must in such a case be transformed into amounts orthogonal to the diagonal. The standard errors of the residuals are then to be determined in order to see if the residuals are significant.

It is first necessary, however, to determine the standard errors of the residual coordinate errors after the adjustment of the calibration. For this purpose, the procedure used for the calibration and for the computation of the residuals must be carefully taken into account. We assume here that the procedure derived above in this report has been used for the calibration (see paragraph 3). The adjustment has, consequently, been made with the aid of one point in the center of the photographs and four points on a circle around this point with the radius r. The residuals are then computed from the expressions:

$$v_x = -\,dx_o - \frac{x}{c}dc + y\,d\kappa + \left(1 + \frac{x^2}{c^2}\right)c\,d\varphi - \frac{xy}{c}d\omega - dx$$

$$v_y = -\,dy_o - \frac{y}{c}\,dc - x\,d\kappa + \frac{xy}{c}\,d\varphi - \left(1 + \frac{y^2}{c^2}\right)c\,d\omega - dy$$

In order to determine the standard errors of the residuals, the general law of error propagation is applied to these expressions.

$$Q_{v_xv_x} = Q_{x_ox_o} + \frac{x^2}{c^2}Q_{cc} + y^2Q_{\kappa\kappa} + \left(1 + \frac{x^2}{c^2}\right)^2 c^2\,Q_{\varphi\varphi} + \frac{x^2y^2}{c^2}Q_{\omega\omega} -$$

$$-\,2\left(1 + \frac{x^2}{c^2}\right)c\,Q_{x_o\varphi} + 1$$

$$Q_{v_y v_y} = Q_{y_o y_o} + \frac{y^2}{c^2} Q_{cc} + x^2 Q_{\kappa\kappa} + \frac{x^2y^2}{c^2} Q_{\varphi\varphi} + \left(1 + \frac{y^2}{c^2}\right)^2 c^2 Q_{\omega\omega} +$$

$$+ 2\left(1 + \frac{y^2}{c^2}\right) c\, Q_{y_o\omega} + 1$$

Next, the weight- and correlation-numbers are to be substituted by the expressions from the solution of the normal equations. These have been given above in paragraph 3 (15). Hence, we find after some computations:

$$Q_{v_x v_x} = \frac{10x^4 + 10x^2y^2 - 5x^2r^2 + 3y^2r^2 + 16r^4}{12r^4}$$

$$Q_{v_y v_y} = \frac{10y^4 + 10x^2y^2 - 5y^2r^2 + 3x^2r^2 + 16r^4}{12r^4}$$

The correlation number is $Q_{v_x v_y} = \dfrac{xy\ (5x^2 + 5y^2 - 4r^2)}{3r^4}$

The standard errors of the residuals are then found from

$$s_{v_x} = s_o\sqrt{Q_{v_x v_x}}$$

$$s_{v_y} = s_o\sqrt{Q_{v_y v_y}}$$

where s_o is the standard error of unit weight of the image coordinates.

From these expressions, the standard error of arbitrary residuals can be found after substituting the coordinates of the location of the residual (Fig. 17).

If the residuals are located along the diagonals of the photograph and the x- and y-directions as usual are parallel to the sides of the photograph, a certain simplification can be made as follows.

It can easily be shown that if the residuals in a point along a diagonal are v_x and v_y the corresponding length of the residual, orthogor ' to the diagonal, is of the form $d = \dfrac{v_x - v_y}{\sqrt{2}}$

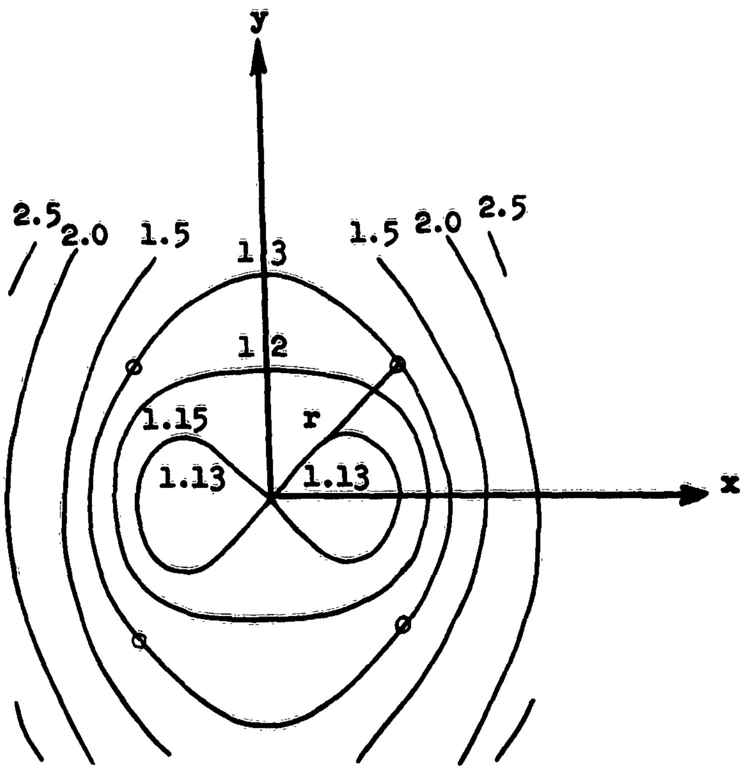

Fig. 17. Graphical representation of s_{v_x} for $s_o = 1$. The corresponding graph for s_{v_y} can be obtained from rotating this Figure through 90^o.

In order to find the standard error of the distance d from the diagonal, the expressions for v_x and v_y above are substituted into the expression d and then the general law of error propagation is applied to the obtained expression. After rather comprehensive computations and after substitution of the expressions (15) for the weight and correlation numbers, the weight number Q_{dd} is found as follows

$$Q_{dd} = \frac{(x^2 + y^2)\ \left\{10(x-y)^2 - 2r_o^2\right\} + 16r_o^2(2r_o^2 + xy)}{24r_o^4}$$

The standard error s_d is then found as usual

$$s_d = s_o \sqrt{Q_{dd}}$$

s_o is the standard error of unit weight of the image coordinates, referred to the adjustment of the circle with the radius r_o.

For an arbitrary point on the corresponding diagonal with the radius r from the principal point, the x- and y-coordinates can be written as $\frac{r}{\sqrt{2}}$. After substitution into the expression Q_{dd} above, we find:

$$Q_{dd} = \frac{r^2}{4r_o^2} + 2$$

Because of the symmetry, this expression must be valid for points on all diagonals with the corresponding radius.

For r_o = 88 mm, we find the following standard errors for other used radii.

$$s_{d_{20}} = 1.41s_o \qquad s_{d_{88}} = 1.50s_o$$

$$s_{d_{41}} = 1.43s_o \qquad s_{d_{106}} = 1.54s_o$$

$$s_{d_{64}} = 1.46s_o \qquad s_{d_{128}} = 1.59s_o$$

For s_o = 3 microns, the standard errors s_d vary between 4.2 and 4.8 microns.

In a problem of this nature, it is always suitable to determine the confidence limits of the residuals. Assuming a normal distribution (at least on the 5-percent level) of the residuals behind the determination of the standard error of unit weight, 4 degrees of freedom, and the 5-percent confidence level, the t-distribution gives the factor $\pm$ 2.8 with which the standard error is to be multiplied in order to find the actual confidence limits. For the average 4.5 microns as standard error, this gives the value 0.012 microns or with some approximation 0.01 mm. In other words, on the chosen level the individual residuals should be of the order of magnitude of 0.01 mm in order to be regarded as caused by sources of errors other than irregular ones. In addition to this criterion, it is possible to use some condition concerning the direction of the residuals. Such a possible criterion will be treated next.

We assume the residual coordinate errors to be determined for points along a diagonal and that the residuals orthogonal to the diagonal are shown. Next, the residuals in the end points of the diagonals are made zero through rotation and translation of the entire point sequence as treated under paragraph 5a. The residuals after this operation are next used for a criterion whether or not there is a significant curvature of the diagonal. Each residual is first multiplied by the distance to the nearest end point, and the sum of the products is computed. The center point in the sequence

is multiplied by both distances to the end points. The sum of the products is evidently an expression for the curvature, the significance of which, however, must be tested in some way. For this purpose, the standard error of the sum of the products will be determined and the deviations of the sum from zero will be tested with respect to the standard error.

We assume the test points to be located symmetrically with respect to the principal point. If the distances of the test points, including the principal point, from the end points of the diagonal are denoted $x_1, x_2, \ldots, x_p$ and the corresponding residuals are denoted $d_1, d_2, \ldots, d_p$ and $d_{-1}, d_{-2}, \ldots d_p$ respectively (see Fig. 18), we have the following expression for the sum of the products S

$$S = x_1(d_1 + d_{-1}) + x_2(d_2 + d_{-2}) + \ldots + 2x_p d_p$$

Assuming the standard error of the residuals d to be at least approximately the same as was shown above (the magnitude $1.5s_o$ is a good average value), the standard error of S can with some approximation be determined from the special law of error propagation:

$$s_S = s_o \sqrt{4.5([xx] + 2x_p^2)}$$

The confidence limits of S can then be found, again with some approximation, with respect to the standard error s_S and with the aid of the t-distribution. If s_o was determined from 4 degrees of freedom, the factor 2.8 is found for the 5-percent level. Consequently, if the product sum S exceeds the value $\pm$ $2.8s_S$, tangential distortion can be regarded to be present. There are some approximations in this derivation, in particular neglected correlations. The confidence level is, further, a choice. Therefore, this criterion should primarily be understood as a guidance. Next, a practical example of the application of the criterion will be shown. From the publication (1), Diagram 13, p. 35, the following example is taken. After a test of an aerial camera Wild R.C. 5a, Aviogon c = 152 mm through photography of a grid on the ground from a high tower, the residual coordinate errors along the two diagonals 108 - 408 and 208 - 308

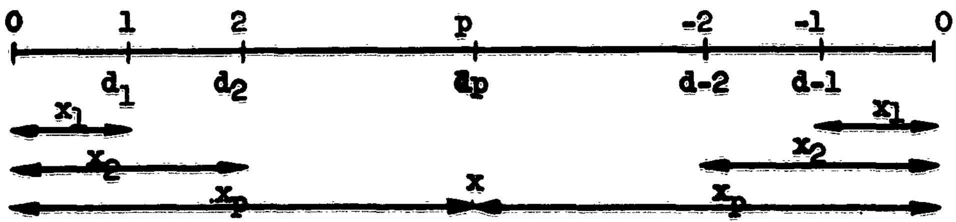

Fig. 18. Schematic illustration to the computation of the sum of the products of the residuals d_1, d_2, etc. and the corresponding distances x_1, x_2, etc.

were computed as shown in the mentioned diagram. The adjustment was made with the aid of the center point 5 and the four mentioned points, located on a circle around the center point with a radius of 118.6 mm in the image. The residual coordinate errors were first transformed into residuals, orthogonal to the diagonals. These residuals are shown in Table XIV. In the same table, the procedure of the translation and rotation of the diagonals for reducing the residuals in the end points of the diagonals to zero is also shown. The principles of this procedure were shown above under paragraph 5a.

Table XIV. Residual Image Coordinate Errors Along Two Diagonals before and after Translation and Rotation of Diagonals

Point	r (mm)	x (mm)	d (micr)	d after transl. and rot. (micr)	Point	r (mm)	x (mm)	d (micr)	d after transl. and rot. (micr)
108	119	0	+2	0	208	119	0	-1	0
107	104	15	+3	+1	207	104	15	+2	+3
106	90	29	+8	+6	206	90	29	-2	-1
105	74	45	+4	+3	205	74	45	+8	+10
104	59	60	+6	+5	204	59	60	+3	+5
103	44	75	+4	+3	203	44	75	+1	+3
102	30	89	-3	-4	202	30	89	+7	+9
101	15	104	+8	+8	201	15	104	+7	+9
5(p)	0	119	-3	-3	5(p)	0	119	+5	+8
401	15	104	+2	+2	301	15	104	+5	+8
402	30	89	+3	+3	302	30	89	+6	+9
403	44	75	+4	+5	303	44	75	-2	+1
404	59	60	-3	-2	304	59	60	-10	-7
405	74	45	+9	+10	305	74	45	+7	+10
406	90	29	+10	+11	306	90	29	-3	+1
407	104	15	-3	-1	307	104	15	+1	+5
408	119	0	-2	0	308	119	0	-4	0

The parameters for the translation and rotation are

for the diagonal 108 - 408: $dr_o = -2$, $d\varphi = \frac{4}{238}$ and

" " " 208 - 308: $dr_o = +1$, $d\varphi = \frac{3}{238}$

From the table: $[xx] + 2x_p^2 = 90428$; $S_{108-408} = +2095$; $S_{208-308} = +6474$

In this case, the standard error of unit weight is $s_o = 4.4$ microns. Hence, $s_S = 2807$ microns. The confidence limits become: $\pm$ 7860. The conclusion is, therefore, that tangential distortion is not regarded to be present in any of the tested diagonals. In Fig. 19, the residuals are shown graphically. According to Table XIV and this figure, there is a tendency of tangential distortion in both diagonals, in particular in 208-308, but the amounts are too small according to the test.

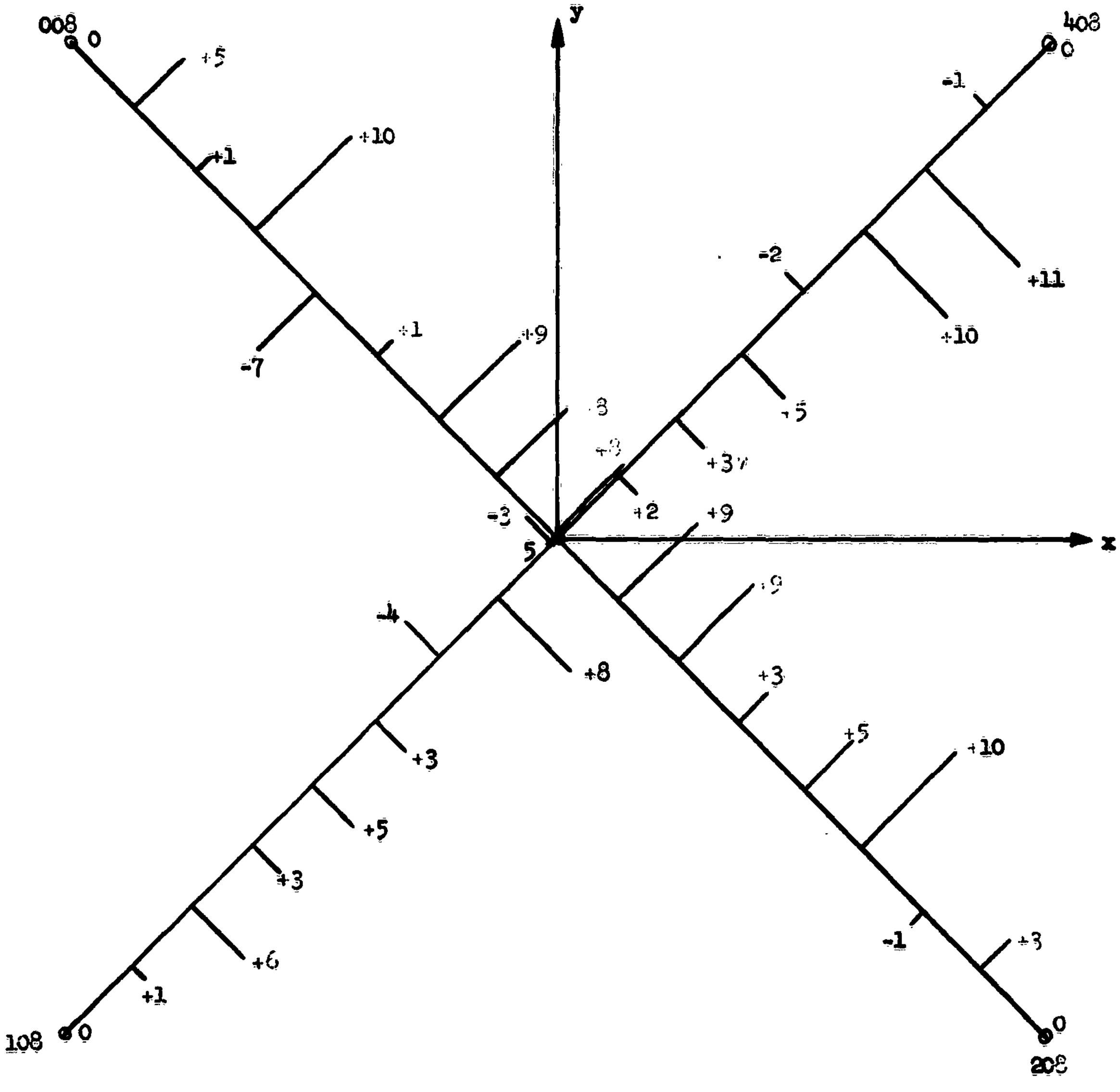

Fig. 19. Residuals along the diagonals, and orthogonal to the diagonals, after adjustment and after making the residuals zero in the points 108-408, 208-308 through translations and rotations of the diagonals. All data in microns.

It should again be emphasized that there are certain approximations in the procedure, which may be regarded as necessary for the practical treatment of the problem. For the practical application of the test for tangential distortion, it is necessary to test at least two sets of diagonals. In the camera calibration, therefore, at least two photographs should be taken between which the camera should be rotated through 45° around the camera axis.

6. Empirical Determination of Correlation between Residual Image Coordinate Errors. For several purposes, the correlation between the residual image coordinate errors after the adjustment of the calibration procedure is of great interest. Two types of correlation can be distinguished, viz. the "algebraical" correlation from the adjustment procedure and the "physical" correlation. The former correlation can be determined theoretically from the basic differential formulas for the residuals in terms of the elements of orientation and the normal equation systems, but the latter correlation must be determined empirically from real residuals in connection with numerical adjustments of the calibration procedure.

Here, the basic expressions for the algebraical correlation will first be derived and then the empirical correlation will be investigated with the aid of a series of performed calibrations of aerial cameras.

The basic differential expressions for the determination of the algebraical correlation are for two points 1 and 2:

$$v_{x1} = -dx_o - \frac{x_1}{c}dc + y_1 d\varkappa + \left(1 + \frac{x_1^2}{c^2}\right)c d\varphi - \frac{x_1 y_1}{c} d\omega - dx_1$$

$$v_{x2} = -dx_o - \frac{x_2}{c}dc + y_2 d\varkappa + \left(1 + \frac{x_2^2}{c^2}\right)c d\varphi - \frac{x_2 y_2}{c} d\omega - dx_2$$

$$v_{y1} = -dy_o - \frac{y_1}{c}dc - x_1 d\varkappa + \frac{x_1 y_1}{c}d\varphi - \left(1 + \frac{y_1^2}{c^2}\right)c d\omega - dy_1$$

$$v_{y2} = -dy_o - \frac{y_2}{c}dc - x_2 d\varkappa + \frac{x_2 y_2}{c}d\varphi - \left(1 + \frac{y_2^2}{c^2}\right)c d\omega - dy_2$$

The correlation between v_{x1} and v_{x2} can be expressed with the aid of the correlation number which is obtained after determining the sum of a cross product between the coefficients of the differentials in all combinations. The corresponding weight and correlation numbers of the elements of orientation have been shown above, (15). We find:

$$Q_{v_{x1}v_{x2}} = 2Q_{x_ox_o} + \frac{2x_1x_2}{c^2} Q_{cc} + 2y_1y_2Q_{\varkappa\varkappa} + \frac{2(x_1^2 + c^2)(x_2^2 + c^2)}{c^2} Q_{\varphi\varphi} +$$

$$+ \frac{2x_1x_2y_1y_2}{c^2} Q_{\omega\omega} - (x_1^2 + x_2^2 + 2c^2) \frac{Q_{x_o\varphi}}{2}$$

$$Q_{v_{y1}v_{y2}} = 2Q_{y_oy_o} + \frac{2y_1y_2}{c^2} Q_{cc} + 2x_1x_2Q_{\varkappa\varkappa} + \frac{2x_1x_2y_1y_2}{c^2} Q_{\varphi\varphi} +$$

$$+ \frac{2(y_1^2 + c^2)(y_2^2 + c^2)}{c^2} Q_{\omega\omega} + (y_1^2 + y_2^2 + 2c^2) \frac{Q_{y_o\omega}}{2}$$

After substitution of the expressions for the weight and correlation numbers from (15), we find:

$$Q_{v_{x1}v_{x2}} = \frac{1}{6r^4} (4r^4 + 10c^4 + 4r^2c^2 + 3r^2x_1x_2 + 3r^2y_1y_2 + 10x_1^2x_2^2 + 5x_1^2c^2 +$$

$$+ 5x_2^2c^2 + 10x_1x_2y_1y_2 - 2x_1^2r^2 - 2x_2^2r^2)$$

$$Q_{v_{y1}v_{y2}} = \frac{1}{6r^4} (4r^4 + 10c^4 + 4r^2c^2 + 3r^2x_1x_2 + 3r^2y_1y_2 + 10y_1^2y_2^2 + 5y_1^2c^2 +$$

$$+ 5y_2^2c^2 + 10x_1x_2y_1y_2 - 2y_1^2r^2 - 2y_2^2r^2)$$

The correlation coefficients can then be found from the ratios:

$$\frac{Q_{v_{x1}v_{x2}}}{\sqrt{Q_{v_{x1}v_{x1}} Q_{v_{x2}v_{x2}}}} \quad \text{and}$$

$$\frac{Q_{v_{y1}v_{y2}}}{\sqrt{Q_{v_{y1}v_{y1}} Q_{v_{y2}v_{y2}}}}$$

The expressions for $Q_{v_xv_x}$ and $Q_{v_yv_y}$ have been derived above (see paragraph 5c).

In these expressions, attention has been paid to the influence of the elements of orientation only. In practice, there are many more sources of errors influencing the residuals of the image coordinates. Therefore, an empirical investigation of the correlation relations between the residuals must be more reliable than the theoretical derivation founded upon the elements of orientation only. Such an empirical determination of the correlation has been made with the aid of residual image coordinate errors after a series of adjustments of camera tests from the air, i.e. under real operating conditions. The primary purpose of the determination of the correlation was to investigate the mutual relations between the residuals in the corners of squares located in different parts of the photographs and of different size. The results of the correlation tests are to be applied to practice for the following tasks.

A regular point (grid) pattern is to be superimposed to a photograph and is to be geometrically connected with the image coordinate system of the photograph (defined by the fiducial mark system). The coordinates of the grid points in the moment of exposure are assumed to be known with high quality and are also assumed to be measured with high quality again in the film when the image coordinates of arbitrary details are to be determined for the photogrammetric procedure. From a comparison between the two sets of coordinates of the pattern points, possible film deformations can be numerically determined and corrections can be applied to simultaneously measured coordinates of image details before these coordinates are used in the photogrammetric procedure. This procedure evidently assumes that the point pattern is so dense that the errors to be corrected within the corresponding areas (squares) are caused by the same sources, primarily film deformations, or that these errors are correlated to a high degree. In this case, the concept of correlation is evidently closely related to the concept of regular errors, in particular the special case constant errors within local areas. The correlation between the residual image coordinate errors and the distances between the corresponding points in the image is, therefore, of basic importance for the determination of the most suitable density of the point pattern. From the point of view of high reliability of the corrections, the point pattern should be very dense but there are also factors involved which require a sparse point pattern (the photographic quality and primarily the time consumption and the costs for the construction and measurements of a dense grid). Therefore, in this case as so often in photogrammetry, a compromise between reliability and economy has to be used.

For further treatment of the problem of density of the point pattern, it is necessary to get some information about the correlation between residual image coordinate errors in aerial photographs on film and in particular about the relation between the degree of correlation and the distances between the actual image points.

As indicated above, such an investigation has to be of empirical character, i.e. the residual image coordinate errors in aerial photographs on film after a determination and correction of the most important regular image coordinate errors (primarily the radial distortion and general affine deformations) must be investigated closer with respect to their mutual correlation. It is of importance that the photographs have been taken under real operational conditions since all possible sources of errors during the photographic procedure must be taken into account. Residual errors after laboratory tests of cameras are also of interest, however, but mainly as additional checks of the correlation pattern from the real aerial photographs. Shrinkage tests of films from laboratory experiments are also of interest if the pattern of residual errors is determined according to an adjustment with the aid of least squares. In this investigation, primarily the residual image coordinate errors as shown in publication (21), p. 555 through 565, Diagrams 6, 9, and 12, will be used. The aerial photographs were in this case taken with three different cameras and on three different rolls of film. The flying altitude was about 5,000 meters (15,000 feet) above the ground and the image coordinate residuals were obtained after adjustments of the space resections according to least squares. The photographs may, therefore, be representative of that type of photography which is of actual interest. For convergent photography, for instance, oblique photographs of suitable test fields (perspective grid arrangements) should be used for the correlation determination.

a. Statistical Distribution of Residuals. First, the normal distribution of the residuals was tested. Although the number of residuals for each image is rather limited (50 residuals in x and y), the images were tested individually. Also, all residuals were tested together. (See the Appendix.)

No detailed description of the tests will be given here since it is satisfactory to show the histograms and the corresponding normal distribution curves in addition to the chi^2 tests of the discrepancies between the two curves (see Figs. 20 through 23). The following results were obtained.

Photograph 1 (Hec 59 01a 08): Well normally distributed.
" 2 (Hec 59 01q 15): Very well normally distributed.
" 3 (Hec 59 02b 05): Not normally distributed.
" 1 + 2 + 3: Normally distributed.

b. Determination of Correlation. The residuals were determined in image points which were located and denoted according to Fig. 24. The point figure was divided into squares of different sizes. The largest square is evidently the figure 22-62-66-26. There are then nine squares the sides of which are the half of the sides of the largest square; there are further 16 small squares.

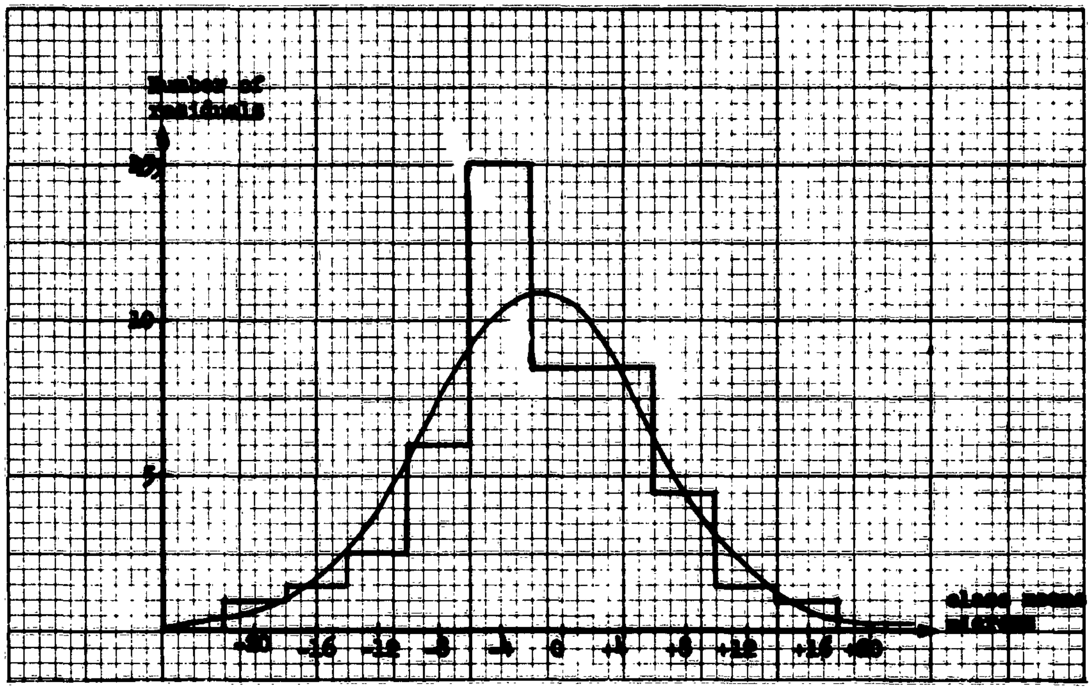

Fig. 20. Histogram and normal distribution curve of residuals from image 1 (Hec 59 01a 08).

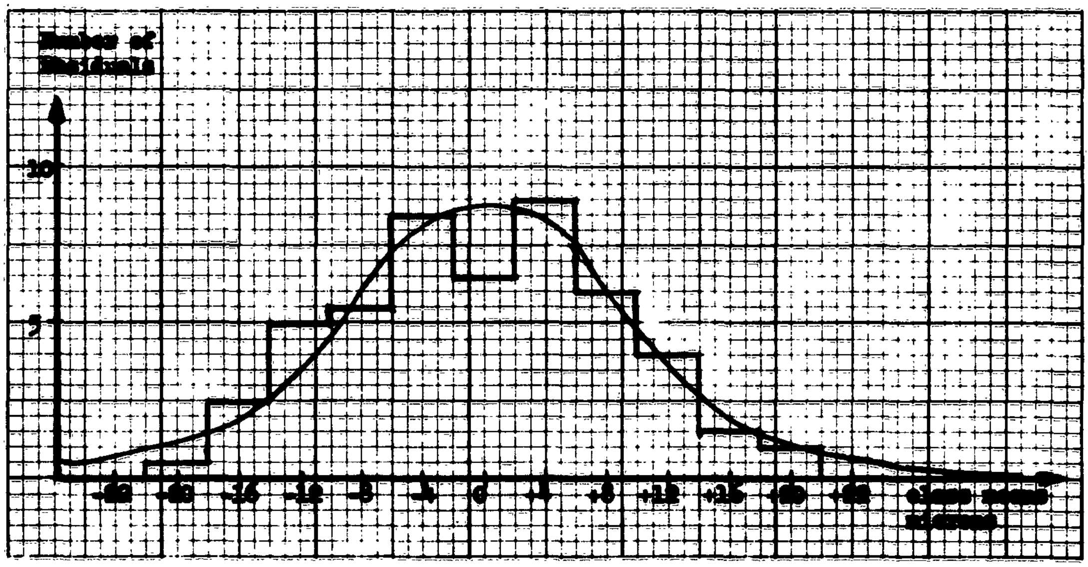

Fig. 21. Histogram and normal distribution curve of residuals from image 2 (Hec 59 01q 15).

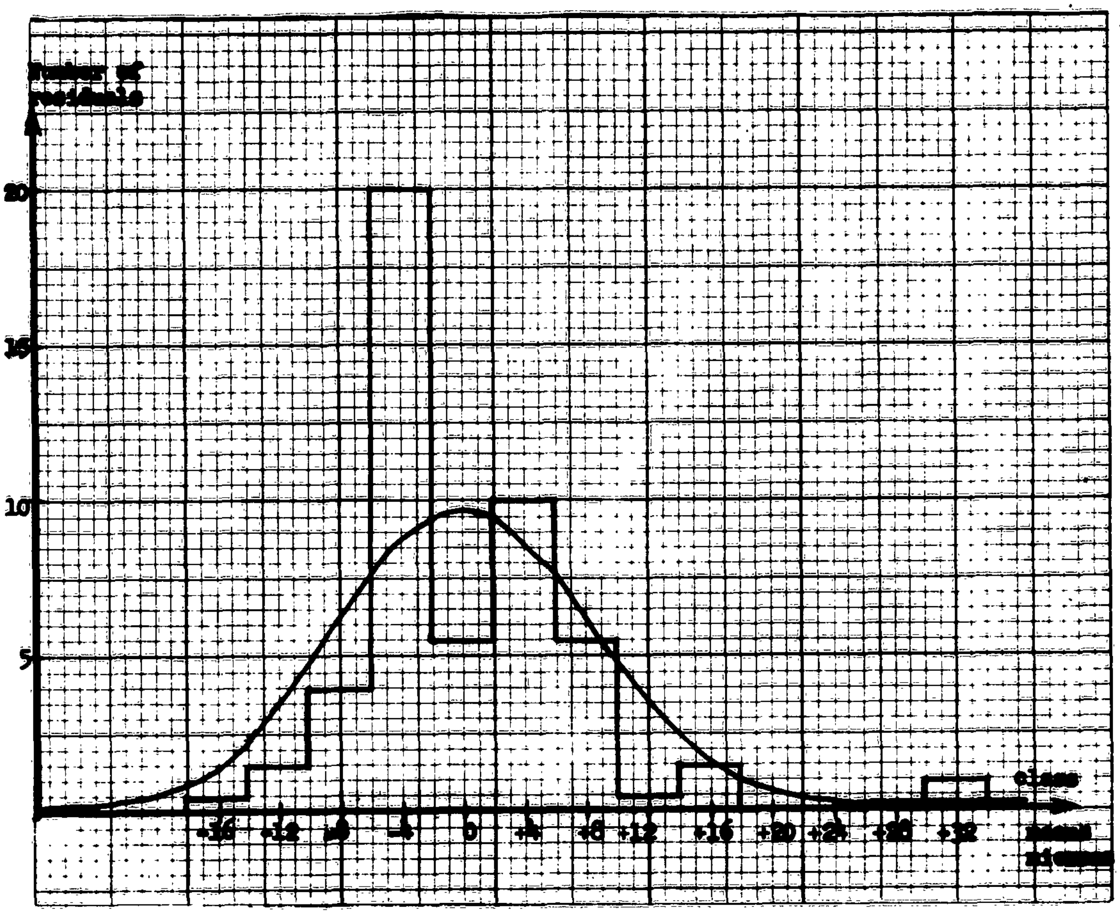

Fig. 22. Histogram and normal distribution curve of residuals from image Hec 59 02b 05 (image 3).

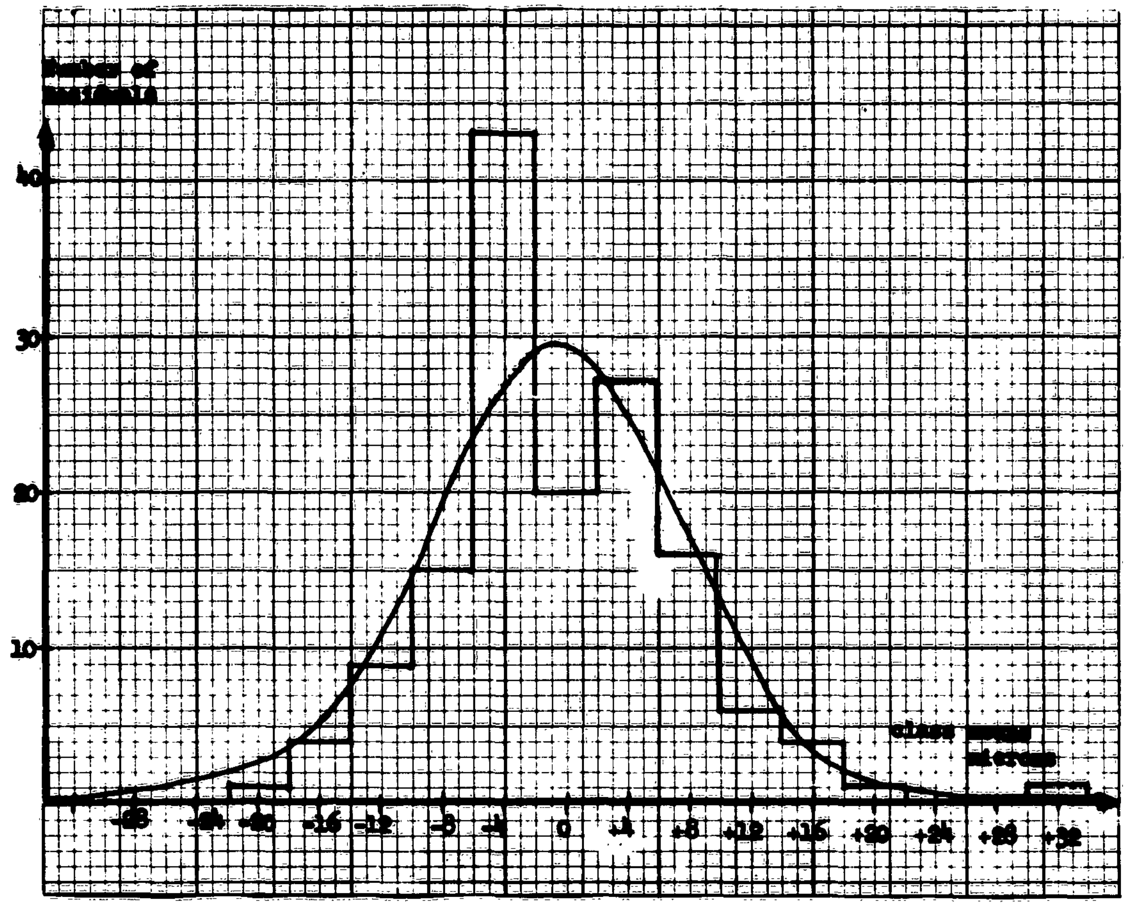

Fig. 23. Histogram and normal distribution curve of residuals from three photographs.

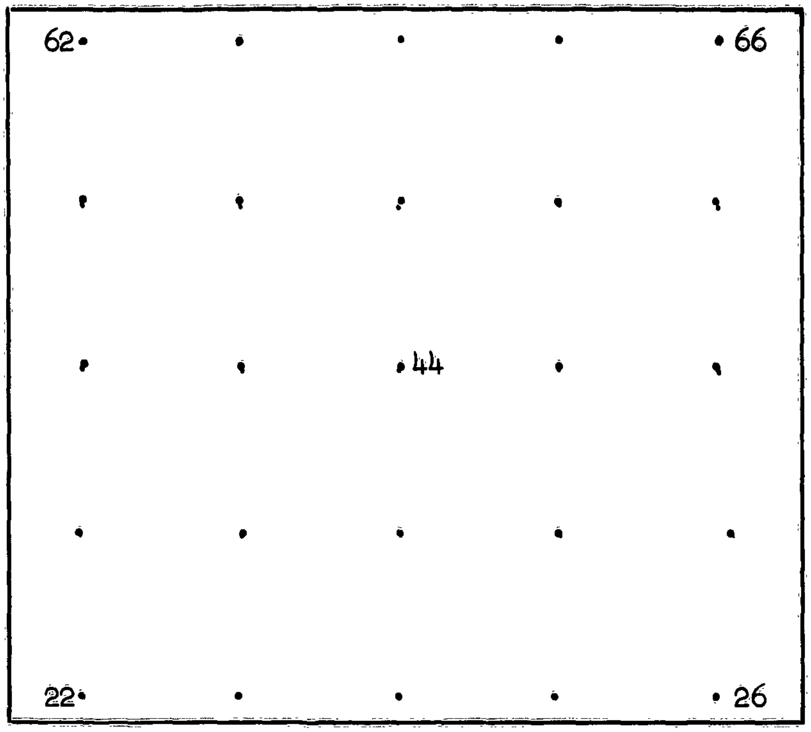

Fig. 24. Location and notation of image points with residual coordinate errors.

The correlation was investigated for the corners of all 26 squares in each of the three photographs or together for 78 squares. For each of the squares, the sum of the products of the residuals in the x-coordinates was computed from all the possible six combinations. Then, this product sum was "normalized" through division by the sum of the squares of the x-residuals. In a similar way, the residuals of the y-coordinates were treated. In formulas, the mentioned empirical expressions for the correlations in the x- and y-directions are defined as follows:

$$c_x = \frac{[r_{xp} r_{xq}] \text{ (p,q cycl.diag.)}}{[r_x r_x]} \qquad c_y = \frac{[r_{yp} r_{yq}] \text{ (p,q cycl.diag.)}}{[r_y r_y]}$$

It should be noticed that these expressions are no correlation coefficients according to usual definitions. If the residuals in the four corners of the squares are equal in magnitude and direction, the expressions above obtain the value $\frac{6}{4}$ or 1.5. Consequently, if the squares become smaller and smaller, the correlation values can be assumed to approach the value 1.5 since the residuals due to film deformations must become more and more equal. The irregular errors

of the measurements will, of course, to a certain extent have influence upon these relations.

The dimensions of the largest square are 200 x 200 mm and of the two other sets of squares 100 x 100 and 50 x 50 mm, respectively. It would, of course, be desirable to have still smaller squares available for these investigations, but so far only the mentioned dimensions are at disposal. It seems possible, however, to use some kind of extrapolation between the smallest square and the limit value 1.5. Further, some additional tests may possibly be applied from residuals in connection with multicollimator tests.

In Table XV, the results of the computations from the residuals of the three test photographs as mentioned above are summarized. Only the absolute values of the computed correlation data are used since the sign of the correlation is of no importance for the actual purpose.

Table XV. Correlation Expressions for Test Photographs

Squares	Photograph 1			Photograph 2			Photograph 3			Average
	x	y	Aver.	x	y	Aver.	x	y	Aver.	
200 x 200 mm	0.32	0.50	0.41	0.11	0.40	0.26	0.41	0.59	0.50	0.39
100 x 100 mm	0.31	0.30	0.30	0.49	0.42	0.45	0.50	0.51	0.50	0.42
50 x 50 mm	0.52	0.44	0.48	0.75	0.71	0.73	0.42	0.61	0.52	0.58

If the three averages are referred to the sides of the squares and the relations are plotted in a diagram, Fig. 25 is obtained. The relation between the correlation amounts and the sides of the squares agree mutually rather well and also with the limiting value 1.5. The analytical expression for the curve has been computed by Dr. D. Harkin as follows:

$$u = 0.0000795v^3 - 0.026025v^2 + 2.2125v + 1.5$$

Evidently, the point from the square side 200 mm is rather weak since there are only six determinations behind the value but this value is of very limited value for the practical application. The important part of the curve and numerical expression refers to square sides less than 50 mm. For the 50-mm side, there are 96 determinations of the correlation amount. In the graphical representation (Fig. 25), the results of a similar correlation investigation from residuals after adjustments of multicollimator tests of aerial cameras have also been shown. The tests were made in using films also. The agreement with the existing curve is evidently rather good.

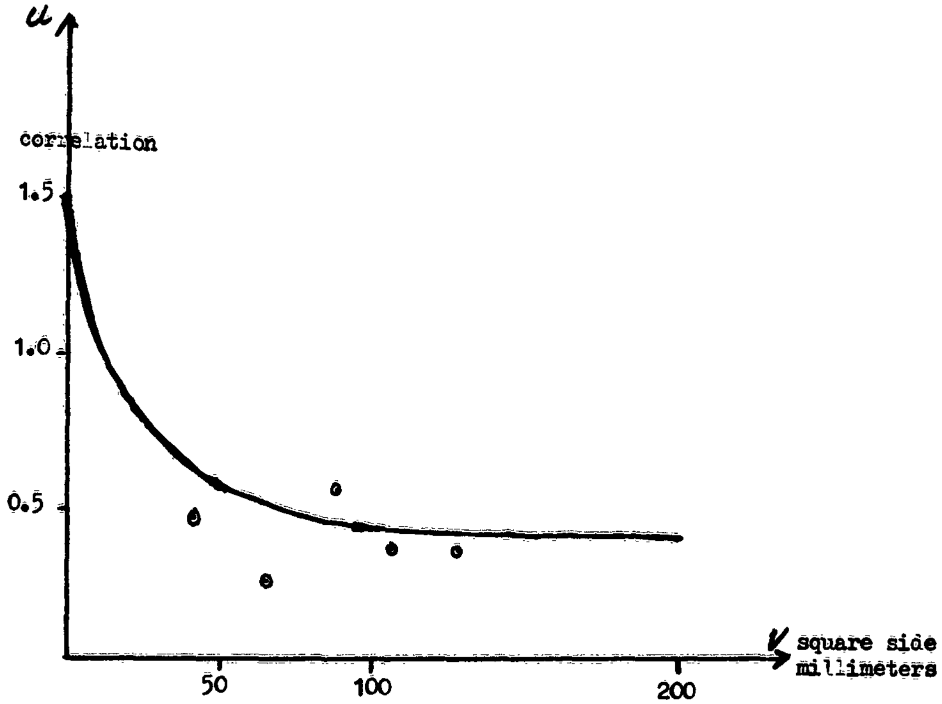

Fig. 25. Graphical representation of the relation between correlation amounts and sides of the squares. The dots are correlation amounts from residuals in film negatives after multicollimator tests.

Hence, it has been found from these empirical investigations that a certain correlation exists between the residual image coordinate errors and that this correlation increases when the distances between the points decrease. A numerical expression for the correlation amounts when the points are located in the corners of squares has also been determined. These results are of great interest for a number of applications. It is possible, as was mentioned above, to determine the necessary density of a grid pattern to be superimposed in connection with the photography in order to correct certain regular errors of the image coordinates as caused by, for instance, shrinkage of the film, etc. In such a case, the best results are probable when the correlation within the squares of the grid reaches a certain level. Evidently, this correlation becomes stronger the smaller the squares are made according to Fig. 25. The choice of the density of the grid must, therefore, be a compromise between the correlation and the economic conditions concerning the manufacture, calibration, and measurements of the grid. For such purposes, the determined correlation amounts are of importance. It must be emphasized, however, that the performed tests are comparatively limited and that there might be differences between different film makes and that the correlation also might be dependent upon the

photographic treatment, in particular the drying process. Further correlation tests are, therefore, much needed.

Among further applications of results of correlation tests, the construction of specifications for distance measurements in photographs and photogrammetric models can be mentioned. For the determination of the accuracy to be expected in the measurements of distances between points, the correlation must be taken into account and attention must evidently be paid to the possible variation of the correlation with the distance itself.

7. Tests of Some Basic Geometrical Qualities of Photographic Material. The basic material in all photogrammetric activity is the photographic negative. In general, photographic qualities such as speed, sensitivity, graininess, resolving power, etc., are very carefully investigated and tested; this is of importance primarily with respect to the interpretative applications of the photograph. The problem of how to define and measure the resolving power of a photograph is, however, not yet completely solved and in particular the relation between this quality and the geometric qualities of the image coordinates is not clear.

Certain of the geometric qualities of the photographic negative material, primarily the shrinkage of different film bases and the flatness of glass plate negatives, have been tested and are still continuously being investigated for the purpose, among other things, of establishing tolerances. There are, however, more geometrical qualities of the negative material which are of basic importance for the photogrammetric application and which have been comparatively little tested. In particular, the flatness of the negative surface in the moment of exposure is evidently of the greatest importance for the accuracy of the image coordinates if the measurements or observations in the image are to be performed orthogonally to the surface. Orthogonal measurements and observations of image coordinates are used mainly in modern photogrammetric instruments, as for instance all stereocomparators and in all instruments with mechanical projection as the Wild and Santoni instruments. There are several factors which can affect the flatness of the negative surface in the moment of exposure and which, therefore, are of importance for the accuracy of the image coordinates in orthogonal measurements. First of all, it is necessary to define and express the concept of flatness clearly. In this respect, there are different expressions in use which sometimes may be very difficult to interprete. It is, for instance, not clear if the expression "flatness of.....microns" means a root mean square value of the elevation differences of the surface or some kind of maximum elevation differences between the lowest and highest points or if the expression refers to deviations from an average plane. It seems necessary that such basic definitions be standardized and that the figures

concerning flatness in each individual case be clearly expressed. It is always important to remember that flatness has to be determined with the aid of measurements and that, consequently, the quality of the measuring device and procedure is of basic importance and must be well defined. The plane, to which the flatness variations are to be referred is also to be determined with measurements and the quality of the measurements will, therefore, also determine the position of this plane and the geometrical quality of the deviations from the plane. A method to determine and express the flatness of a surface without touching the surface (the microscope method) and a well-defined procedure for expressing the flatness and the quality of the measurements has been published in reference (11). Significant deviations from flatness are defined there as confidence limits in terms of the basic standard error of unit weight of the measurements. Attention is paid to the fact that the reference plane also is determined with measurements and, consequently, is affected with errors too.

Concerning flatness of photographic negative material, distinction must be made between glass plates and films. The flatness of the image surface in a glass plate negative is primarily determined by the flatness of the glass plate itself. Further, the thickness variations of the emulsion is of importance and also the location of the image details within the emulsion. Concerning the glass plate, there are sometimes specifications given by the plate manufacturers but these usually refer to the glass plate without emulsion and to measurements made in connection with the selection of the plates. These measurements must for natural reasons be made quickly and with the aid of standardized methods, usually based upon the Newton rings. It is possible and very probable that the emulsion is affected with certain thickness variations and that the tensions of the emulsion may cause certain bendings of the glass plate, depending upon the thickness of the plate. In some photogrammetric organizations (for instance, Institut Geographique National (I.G.N.) in Paris, France), therefore, the glass plates are coated on both sides. Further, it is most important that no additional deformations of the glass plates are introduced in connection with the exposure. Serious defects of photogrammetric cameras have been detected in this respect as reported in reference (12). This is also an illustration of the necessity to test equipment and material under operating conditions before acceptance and application to practice. In regard to the thickness variations of the emulsion, no investigations seem to have been performed or published. Some experiments concerning the location of the image details within the emulsion will be reported below.

Concerning film negatives, there are certain sources of disturbances of the flatness in the moment of exposure which should be much more noticed. First of all, the supporting back can never

be made exactly flat and the deviations from flatness may change considerably with the time and other circumstances as, for instance, temperature, mechanical pressure, etc. The supporting plates of the cameras are usually tested in the laboratory only and there under rather special conditions. The important question is how the flatness is maintained under operating conditions, in other words during the exposure time in the air. It is evidently difficult to test this in the air; laboratory tests where operating conditions are simulated as closely as possible should, therefore, be arranged. Further disturbances of the flatness of the film surface can be caused by variations of the thickness of the film base and the emulsion. Finally, the location of the details within the emulsion is also of importance as already mentioned above in connection with the glass plate. In particular, the thickness variations of the films and emulsion are of interest. No investigations seem to have been made or reported. Therefore, a series of tests of the variations of the thickness of some different films of different manufactures has been made and the results will be briefly reported below. Further, some investigations concerning the distribution of the details within the emulsion will also be reported.

a. <u>Investigations of Thickness Variations of Aerial Films</u>. The thickness variations of the films were measured with a measuring gauge where readings could be made in units of 1 micron (0.001 mm). The gauge was placed in a stable stand which was fastened on a surface plate. On the same surface plate, a glass plate of high flatness was placed and fastened. The <u>film</u> was moved over the glass plate into well-defined positions and the surface was touched with a measuring gauge with a constant pressure. In certain points, a great number of replicated settings were made in order to determine the precision of the measurements. The points in the films where the measurements were made were defined with the aid of holes in masks of paper which were fastened on the film. In certain cases, particularly when unexposed film was to be measured, the film was kept flat with the aid of a steel wire frame. The locations and notations of the points for the thickness measurements are shown in Fig. 26.

Eighteen samples from seven different makes of 9- by 9-inch aerial camera film were tested. Four samples were taken from undeveloped and 13 samples from developed film. One sample was measured before and after the development procedure. The measurements were made according to four different systems (Fig. 26):

(1) <u>Measurements in 25 Regularly Located Points</u>. The points were measured in three independent series. In each series, one setting was used per point. The standard deviations of the settings and the root mean square values of the thickness variations were computed.

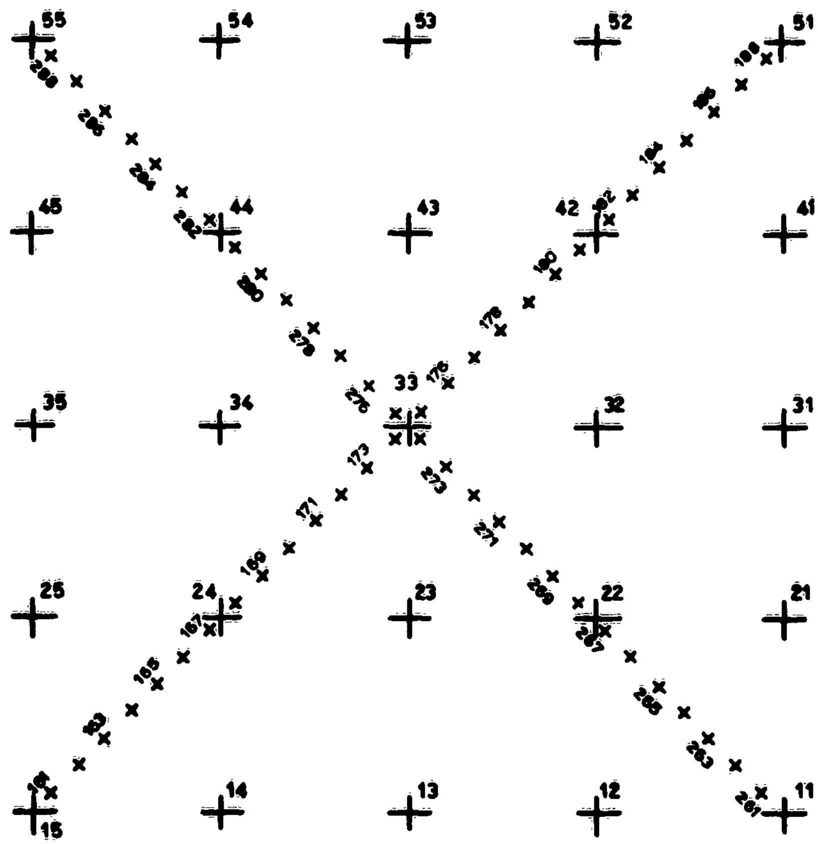

Fig. 26. Locations and notations of the points in which measurements of the thickness variations were made.

(2) Measurements along the Diagonals. Here, three independent series of measurements were also made. The standard deviation and the root mean square value of the thickness variations were computed.

(3) Measurements along One Diagonal Only, but Using Five Settings in Each Point. This experiment was made in order to test the periodicity of the thickness variation.

(4) Measurements in a Great Number (about 500) of Regularly Located Points to Construct a Contour Line Diagram of Thickness Variations. In each point, one setting was made. About 25 points were independently checked with independent, replicated measurements.

(5) Results. The results of the tests can be summarized as follows. The procedure of measurements gave a high precision. The standard deviation of one measurement, computed according to usual formulas, was found to be of the order of magnitude 0.5

to 1.0 micron. For each measured film, the average of the thickness was determined from the measurements in all points. The deviations between the average thickness of each film and average of the measurements in each point were determined; the root mean square value of the deviations was then determined. These root mean square values are used as expressions for the variation of the thickness. For the 13 samples where measurements were made in 25 points, the root mean square values varied between 0.8 and 2.1 microns. Samples taken within the same film roll showed approximately the same thickness variations. No significant difference could be found between undeveloped and developed film. From measurements along the diagonals, periodic variations in the thickness were found. From special measurements of the thickness variations along the diagonals in many points (see Fig. 26), considerable variations of the thickness were found, which did not show up in the 25 point-pattern because of the periodic variations.

From the contour-line diagram of one of the samples, the periodic variation is clearly visible (see Fig. 27). In the same sample, the diagonals were measured. In Fig. 28, the results of the measurements along the diagonal are shown. In the same figure, the thickness variations along the diagonal from the contour line diagram (Fig. 27) are also shown. The agreement between these two completely independent sets of measurements is evidently very good.

(6) Summary. The tests have proved a certain variation of the thickness of the films. There is a considerable difference from one film make to another. The thickness variations seem to be of a periodic nature similar to "waves" in the length direction of the film rolls. The period is of the order of magnitude of 40 mm, and the differences in thickness have been found to amount to about 0.01 mm or more. The reasons for the variations can possibly be found in irregularities of the machine. The average thickness of the tested films is 0.13 to 0.15 mm.

More tests of this type should be made in order to determine if specifications for the thickness variations are necessary. Under all circumstances, such variations will have influence upon the geometrical accuracy of the image coordinates after orthogonal measurements. The influence will increase with the distance of the actual image coordinate from the center of the photograph; this is obviously one of the reasons for the weight variation of image coordinates within the photographic image.

b. Investigations of Location of Image Details in Emulsion. The thickness of the emulsion in aerial films is about 0.01 mm. The location of the details within this range is evidently of certain importance for the determination of the image coordinates in instruments with orthogonal observation. If there are considerable

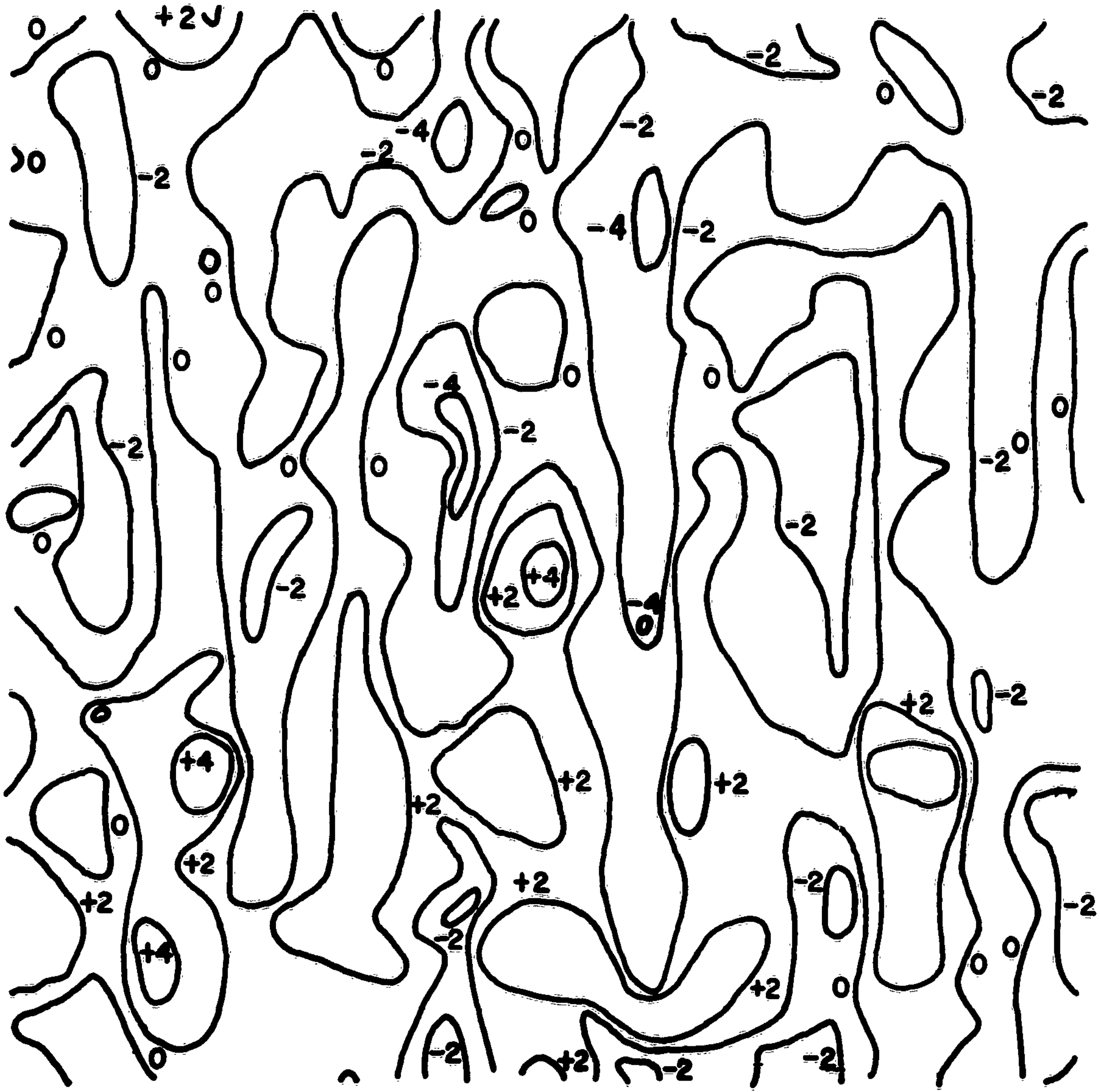

Fig. 27. Contour line diagram of thickness variations of aerial film (figures in microns).

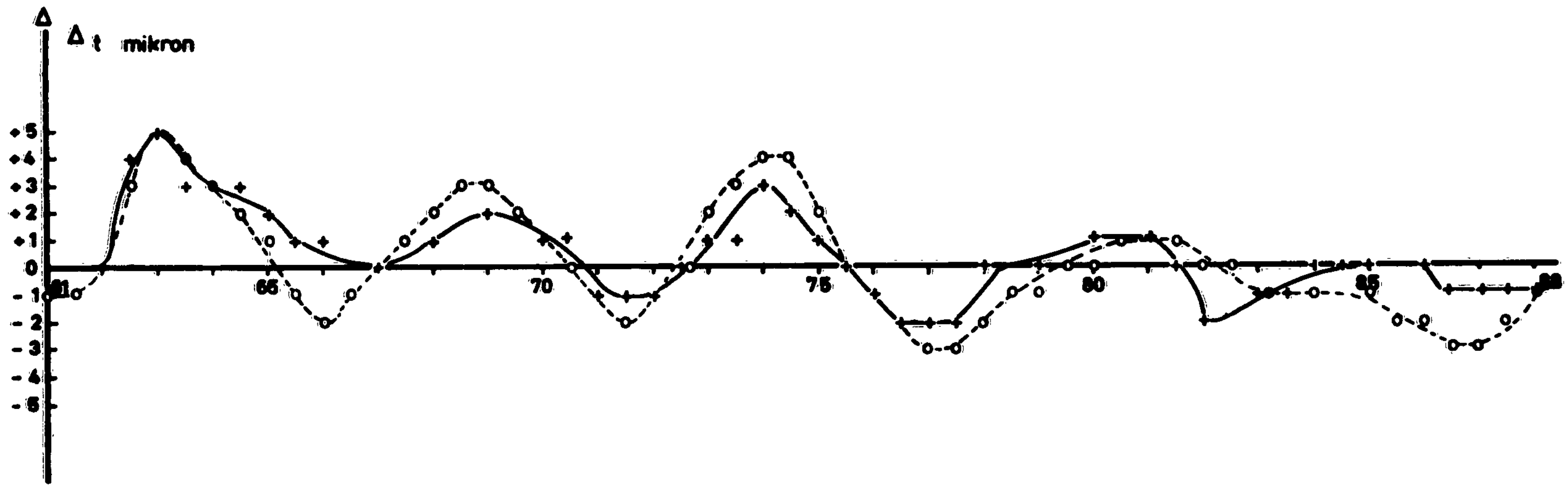

Fig. 28. Thickness variations along a diagonal from Fig. 27 and from direct measurements (the dashed curve).

variations in the location, there must be radial displacements of the individual details depending upon the distance from the center of the image and, consequently, influences upon the image coordinates. A complete treatment of the problem concerning which factors affect the location of the details in the emulsion must certainly be very comprehensive. So far, no experiments with respect to photogrammetric principles seem to have been made or reported. It seems that primarily the exposure time and the developing time may be of importance and that, consequently, experiments should be made in order to see if there is some relation between these factors and the distribution of the silver grains within the emulsion. Some empirical experiments have been made in connection with these investigations of the basic accuracy of image coordinates, and a brief summary of the results will be made. The experiments have been reported in reference (13).

The principles of the experiments are founded upon repeated photography of high-contrast signals with a phototheodolite using seven different exposure times. Three series were photographed without filter and one series with a yellow filter. The three first series were developed during three different times. Glass plates were used in order to facilitate the detaching of the emulsion. The emulsion was then cut in thin slices (about 5 microns thick) orthogonally to the surfaces and through the images of the signals. The distribution of the silver grains in the image was then determined through measurements in a microscope. The results of the measurements were treated statistically, and the relations between the factors of illumination, development, and filter on one hand and the point of gravity of the grains were determined.

From these preliminary investigations, it was found that the exposure time has considerable influence upon the location of the details within the emulsion. For a short exposure time, the same details were found to be located closer to the surface of the emulsion than for longer exposure times. The light intensity is also of importance for the location. The found variations were about 3 to 4 microns. Concerning the development time, no significant influence could be found. The filter caused a minor translation of the details toward the surface of the emulsion, less than one micron only. It must be emphasized that these investigations are of introductory nature and that they have been made primarily in order to find a suitable procedure for further investigations. The conclusions are that the applied procedure works well. It is important to mention that glass plates seem to be necessary for the tests because of the difficulties in detaching the emulsion from the film base. The numerical results of the tests indicate that there may be considerable influence upon the location of the image details within the emulsion from the light intensity at the exposure and that, consequently, another reason for the weight variation of the image coordinates has been found.

8. Tests of Central Projection in Diapositive Printers. The central projection of a diapositive projection printer can be investigated with the same procedure as has been applied above to the aerial camera. In particular, if the printing device corrects certain regular errors of the photographs (as for instance radial distortion), it is desirable to test the results of the printing in order to determine that the actual regular errors really become corrected. It is also suitable to determine the basic accuracy of the image coordinates of the diapositive and of the regular errors. Sometimes, dodging procedures are also applied in order to adjust density variations. It seems important to determine that such procedures do not change the basic geometrical qualities of the diapositives. The most effective and convenient procedure for such tests is the grid method. A grid of high and known geometrical quality is imaged through the actual device, and the image coordinates of the projected grid (diapositive) are measured in a comparator, also of high and known geometrical quality. The formula systems of the grid method can then be directly applied to the computations. The results of the procedure shall primarily be determinations of the ordinary six elements of orientation between the original grid and the copy, radial distortion, and estimation of the standard error of unit weight. Affine deformations and tangential distortions can be determined as shown above. Investigations of possible weight variations within the diapositive can also be made. A series of experiments in a diapositive printer has been made and a summary of the results is given here.

An ordinary 9- by 9-inch grid on glass with 20-mm spacing to be used for tests and adjustments of photogrammetric instruments was copied by contact printing on a glass diapositive. The coordinates of the copy were measured in a Mann comparator, the geometrical quality of which had been tested carefully and where the standard error of unit weight of the coordinate measurements can be expected to be about 1.3 microns. The grid was then reduced in the reduction printer to be used for the experiments. The grid was first printed in instrument without correction devices for radial distortion. The printed copy was measured in the Mann comparator, and the grid method was applied for the determination of the geometrical qualities of the central projection. Some small radial distortion tendencies were found, but these were not significant with respect to the standard errors of unit weight. The distortion and the standard errors of unit weight are shown in Fig. 29. The standard error of unit weight is about 2 microns. This indicates a very good quality of the printing and measuring procedure. Another print was made, including the correction device for Metrogon radial distortion. The results of the computations are shown in Fig. 30 where the radial distortion curve of the camera lens is also reproduced according to available information. The deviations between the two curves are hardly significant with respect to the standard

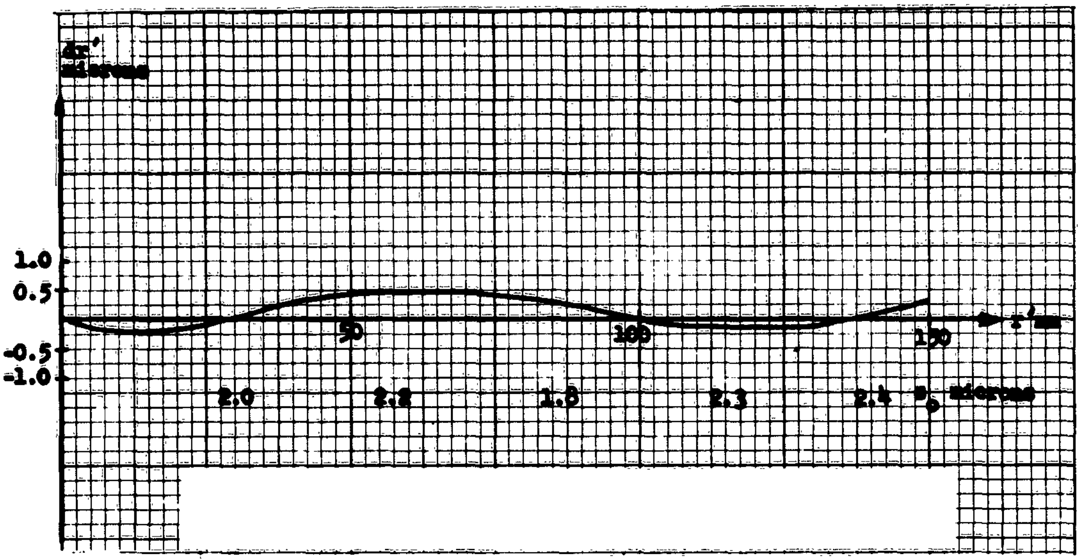

Fig. 29. Reduction printer tests. Zero-distortion and standard errors of unit weight. The obtained distortion amounts are not significant.

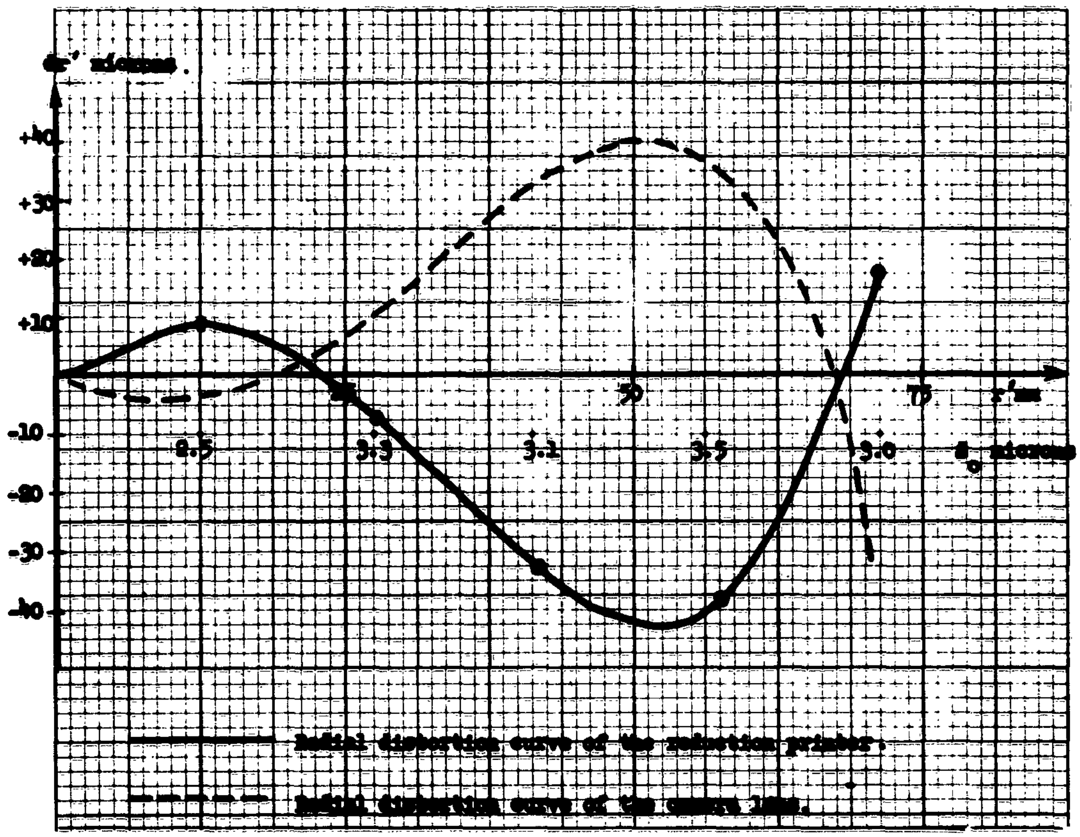

Fig. 30. Reduction printer tests. Radial distortion, Metrogon, and standard errors of unit weight.

error of unit weight shown in the figure. The standard error of the radial distortion is about half of the standard error of unit weight, and about the same amount must be assumed also in the lens distortion curve. A third example is shown in Fig. 31. The correction device of the radial distortion of a Superaviogon lens was used in connection with the printing. Two independent measurements and computations were made. The agreement between the two results is good, and the comparison with the nominal curve of the lens also indicates a rather good agreement. It is apparent from the standard errors of unit weight as shown in the Figs. 29 through 31 that the devices for the correction of the radial distortion introduce certain disturbances. This is to be expected since such aspherical devices are very difficult to manufacture perfectly.

In this experiment also, the corrections to the individual elements of the orientation of the copy were computed together with their standard errors. The results are shown in Table XVI.

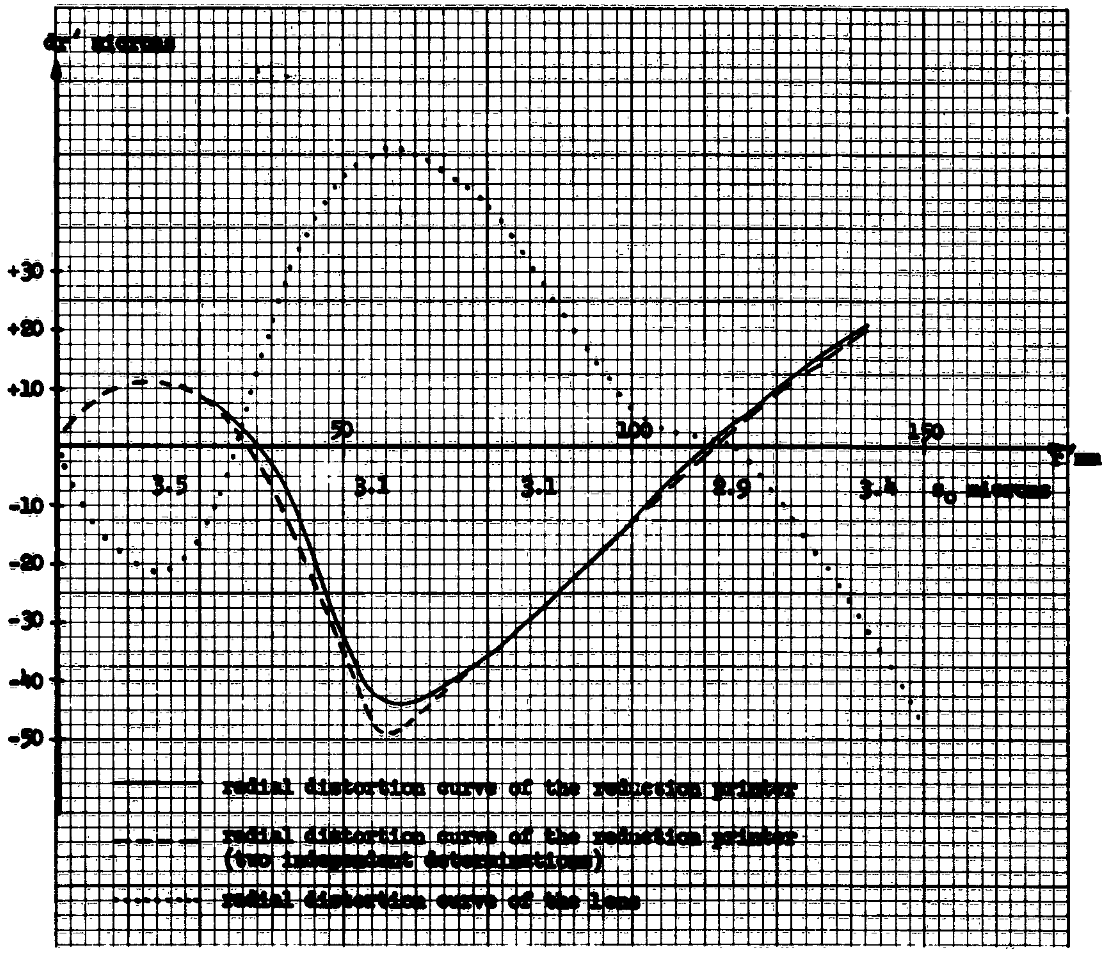

Fig. 31. Reduction printer tests. Radial distortion, Superaviogon, and standard errors of unit weight.

Table XVI. Corrections and Standard Errors of Elements of Orientation

Circle	dx_o	dy_o	$d\varkappa$	$d\varphi$	$d\omega$
	(microns)		(cc (centesimal seconds))		
1 $s_o = 4.7$ s	-73 126	+47 126	-36 53	-252 518	-181 518
2 $s_o = 3.1$	-33 22	+78 22	-6 17	-155 85	-21 85
3 $s_o = 3.1$	+8 10	+4 10	-6 11	+10 38	-11 38
4 $s_o = 2.9$	+5 6	-2 6	-2 8	-1 20	+10 20
5 $s_o = 3.4$	+4 0.5	+1 0.5	-6 8	-2 15	+12 15

To determine whether or not the corrections are significant, the t-test is applied. For 4 degrees of freedom and on the 5-percent level, the factor t_p is 2.8. In no case, is a significant correction found on this level.

In summary, the test procedure has been found to be convenient and reliable and to give good information about the geometrical quality of the printing procedure, which is a very important detail in the photogrammetric procedure.

III. DISCUSSION

The investigations reported in this paper have been primarily concerned with the determination of the geometrical quality of the image coordinates in pictures from aerial cameras. The quality has been defined with respect to the conditions of the central projection and has been expressed in terms of regular and irregular errors in the positions of image details in comparison with the ideal positions according to the central projection for a certain set of elements of the interior orientation. The method of least squares has been applied throughout. The regular errors have primarily been radial distortion and affine deformations but tangential distortion

has also been treated, primarily in order to determine whether or not this type of distortion is present in the results of a calibration procedure. The irregular errors have been estimated in connection with the adjustment procedure. The possible weight variation of the image coordinates as a function of the positions in the image has been investigated. The weight distribution as determined through earlier experiments with aerial photographs has been confirmed. Suitable forms for the computations have been worked out and have been applied to practice. Some relations between the geometrical accuracy of the image coordinates and the photographic resolving power have been studied empirically, and a certain correlation has been found. The well-defined procedure for the calibration of the aerial camera and the photographs has made determination of the accuracy of the elements of the interior orientation possible. The accuracy is expressed in terms of standard errors and as a function of the standard error of unit weight of the image coordinates. It is now possible to establish tolerances for the regular errors of the elements of the interior orientation and also for the standard error of unit weight as confidence limits for certain confidence levels. In this way, a well-defined procedure for tolerance tests of aerial cameras and photographs in connection with the delivery and during practical work can be worked out. It is necessary, however, to determine the standard error of unit weight of the image coordinates more completely through repeated tests with different cameras according to the principles derived and under operating conditions.

Furthermore, some questions concerning the correlation between residuals of image coordinate errors have been treated and, finally, some empirical tests of geometrical qualities of the photographic negative have been reported. These tests have been made primarily in order to find an explanation of the considerable weight variations of the image coordinates. It is evident that much more research should be devoted to these basic problems in order to determine the geometrical quality of the image coordinates under operating conditions. All photogrammetric activity is founded upon such data.

IV. CONCLUSIONS

9. Conclusions. Based on the results of the performed theoretical and practical investigations as reported above, it can be concluded that:

a. The method of least squares is of great value for the calibration of aerial cameras in multicollimators and for additional tests of aerial photographs after photography of regular test fields.

b. The basic geometrical quality, expressed as standard errors, of the elements of the interior orientation, of regular errors of the image coordinates, and of other functions of the image coordinate measurements can be expressed uniquely in terms of the standard error of unit weight of the image coordinates.

c. The basic geometrical quality of image coordinates in glass plates and in films have proved to be approximately identical. The standard error of unit weight of the image coordinates increases with the radius from the principal point. The average was found to be about 3 to 4 microns.

d. Among the reasons for the weight variations of the image coordinates are variations of the resolving power, lacking flatness of the image plane (caused by variations of the thickness of films and emulsions among other things), and varying location of the details within the emulsion.

e. More attention should be paid to the basic physical qualities of the image material.

f. Grid test fields should be arranged in the terrain for tests of the basic geometrical qualities of vertical and oblique photographs from different flying altitudes.

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APPENDIX

TESTS OF ASSUMED NORMAL DISTRIBUTIONS

Abstract

In theory of errors for measurements the normal distribution (normal law, error law, Gaussian law) is of importance. As a matter of fact, the results of an adjustment procedure are dependent on whether the errors of the fundamental measurements are normally distributed or not. The laws for error propagation from the observations to various functions and various significance and confidence tests are also usually founded upon the assumption of a normal distribution. In reports on photogrammetric research, the distribution of the errors, discrepancies, or residuals is seldom given and statistical tests of the normal distribution are hardly ever to be found in photogrammetric literature. An example of such a test will be shown below. The procedure is borrowed from H. Cramér: Mathematical Methods of Statistics, Princeton 1946. The practical example is taken from actual photogrammetric research concerning fundamental operations.

The principles of the test

The actual errors, discrepancies, or residuals can be regarded as a sample from a population, the distribution of which is expected to be normal. This hypothesis is to be tested.

The sample cannot be expected to be exactly normally distributed. There will always be deviations from the theoretical normal distribution. These deviations must be used for the analysis, and certain limits for the deviations must be applied. The chi-square test is generally used for such purposes and will be discussed below. The limits must be determined in relation to certain predetermined probabilities (levels). If the upper limit is exceeded, the normal distribution hypothesis is rejected. On the other hand, very small differences between the theoretical and practical distributions (below the lower limit) cannot be accepted, for obvious reasons.

For the test, first the actual statistical material is grouped, see the table below. The interval which contains all values is divided into a certain number of class intervals. It is desirable that about ten values fall within most of the class intervals and the number of classes should if possible not be less than ten. Sometimes, classes can be pooled in order to increase the number of values. The number of class intervals is denoted k. The class mean (class midpoint) of the class i is denoted t_i and the corresponding

number of values (the class frequency) f_i. Evidently, the sum [f] is the number n of all values. The mean value of the sample is found as

$$\bar{x} = \frac{[ft]}{n}$$

The standard deviation s is found from

$$s = \sqrt{\frac{[vv]}{n-1}}$$

where $[vv] = [ft^2] - \frac{[ft]^2}{n}$

The <u>class limit</u> between the class intervals i-1 and i is denoted b_i. Next the expressions $\frac{b_i - \bar{x}}{s}$ can be computed for all class limits. This expression means the deviation between the class limit b_i and the average $\bar{x}$ of the sample expressed in units of the standard deviation s. The expression is denoted <u>standardized class limit</u>. In the practical computation of the standardized class limits, at least two decimals usually are required. The standardized class limits are then used for the determination of the ideal number of values that should fall within the class intervals if a theoretically strict normal distribution were present. Consequently, the mathematical expression for the normal distribution must be used for further calculations.

If the class limits of the class interval i are denoted a and b, the number of values that theoretically should fall within this interval can be expressed as the product np_i where n is the total number of values in the sample and p_i is a probability which can be determined from the normal distribution function as follows:

$$p_i = \frac{1}{s\sqrt{2\pi}} \int_a^b e^{-\frac{(t-\bar{x})^2}{2s^2}} dt = \frac{1}{\sqrt{2\pi}} \int_{\frac{a-\bar{x}}{s}}^{\frac{b-\bar{x}}{s}} e^{-\frac{\lambda^2}{2}} d\lambda = \varphi \frac{(b-\bar{x})}{s} - \varphi \frac{(a-\bar{x})}{s}$$

This function is tabulated in most textbooks on statistics and theory of errors. From the tables, the values of p_i can be determined for each class interval as the differences between the upper and the lower limits. The theoretically correct number of values which should fall within the actual class interval for a strict normal distribution is then found as np_i.

Next the differences between the theoretical class frequencies and the actual class frequencies are computed as $f_i - np_i$.

Further, the sum $\sum \frac{(f_i - np_i)^2}{np_i}$ is computed. This function expresses the standardized sum of the squares of the differences between the actual distribution and a theoretical normal distribution and shall have a chi-square distribution if the assumed normal distribution of the sample is present. The sum is, therefore, compared with the corresponding value of the chi-square distribution for a certain number of degrees of freedom and for a certain level. In other words, the goodness of fit between the ideal normal distribution and actual distribution of the sample is tested with respect to the differences between the two distributions as expressed by the last function above. If the differences are too large, the normal distribution hypothesis cannot be accepted. If the agreement is too good, the reliability of the sample distribution can be doubted from other points of view. The levels for both cases are usually chosen 5 and 100 - 5 = 95 percent, respectively. The degrees of freedom are in this case determined as k-3, where k is the number of class intervals. From a table of the chi-square distribution, which also is to be found in most textbooks on statistics, the values for the percentages 5 and 95, with k-3 degrees of freedom, are determined and computed sum as indicated above. If the sum is located between the two values from the chi-square distribution, the hypothesis that the sample distribution is normal can be accepted.

Finally, a histogram should be made from the distribution of the sample. In the histogram, the corresponding normal distribution frequency curve can be demonstrated simply, but with a certain approximation, through plotting the computed values of the np_i.

A complete practical example of the computations is demonstrated below together with the histogram and normal distribution curve. The example refers to a published set of figures, see Table 1, page 8 in a communication (2) to the International Congress of Photogrammetry in London 1960, Commission I. The figures are the differences in the x- and y-coordinates from two independent determinations of control points on the ground for tests of aerial cameras from high towers.

The complete computation procedure according to the principles described above is demonstrated in the following table. In the diagram, the actual histogram and normal distribution curve are demonstrated.

The chosen example is not entirely perfect since there are comparatively few values. It is, however, of importance to use a published set of figures so that all details of the procedure can be

checked by those who may be interested. In the actual case, the normal distribution of the difference is also of importance for the camera tests.

Sometimes, the skewness and the excess of a sample are of importance in connection with a normal distribution test. See for instance reference 1.

References:

1. Cramér, H.: Mathematical Methods of Statistics, Princeton, 1946.

2. Hallert, B.: Results of Practical Investigations into the Accuracy of Aerial and Terrestrial Photographs. Communication to Comm. I at the Int. Congress of Photogrammetry, London 1960.

Computations for a normal distribution test

Class means t (mm)	Number of values f	ft	ft^2	Class limits b (mm)	Standardized cl. limits $(b-\bar{x}):s$	p	np	f-np	$\frac{(f-np)^2}{np}$
10	1	10	100						
				9	3.33				
8	0	0	0						
				7	2.59	0.0048	0.4	0.6	0.90
6	2	12	72			0.0274	2.4	-0.4	0.07
				5	1.85				
4	7.5	30	120			0.1013	8.9	-1.4	0.22
				3	1.11				
2	21	42	84			0.2222	19.6	1.4	0.10
				1	0.37				
0	20.5	0	0			0.2886	25.4	-4.9	0.95
				-1	-0.37				
-2	29	-58	116			0.2222	19.6	9.4	4.51
				-3	-1.11				
-4	5	-20	80			0.1013	8.9	-3.9	1.71
				-5	-1.85				
-6	2	-12	72			0.0274	2.4	-0.4	0.07
				-7	-2.59	0.0048	0.4	-0.4	0.40
-8	0	0	0						
				-9	-3.33				
-10	0	0	0						
	88	+4	644				88.0		8.93

$\bar{x} = \frac{4}{88} = 0.05$ mm

Degrees of freedom = 9-3 = 6

$s = \sqrt{\frac{644}{87}} = 2.71$ mm

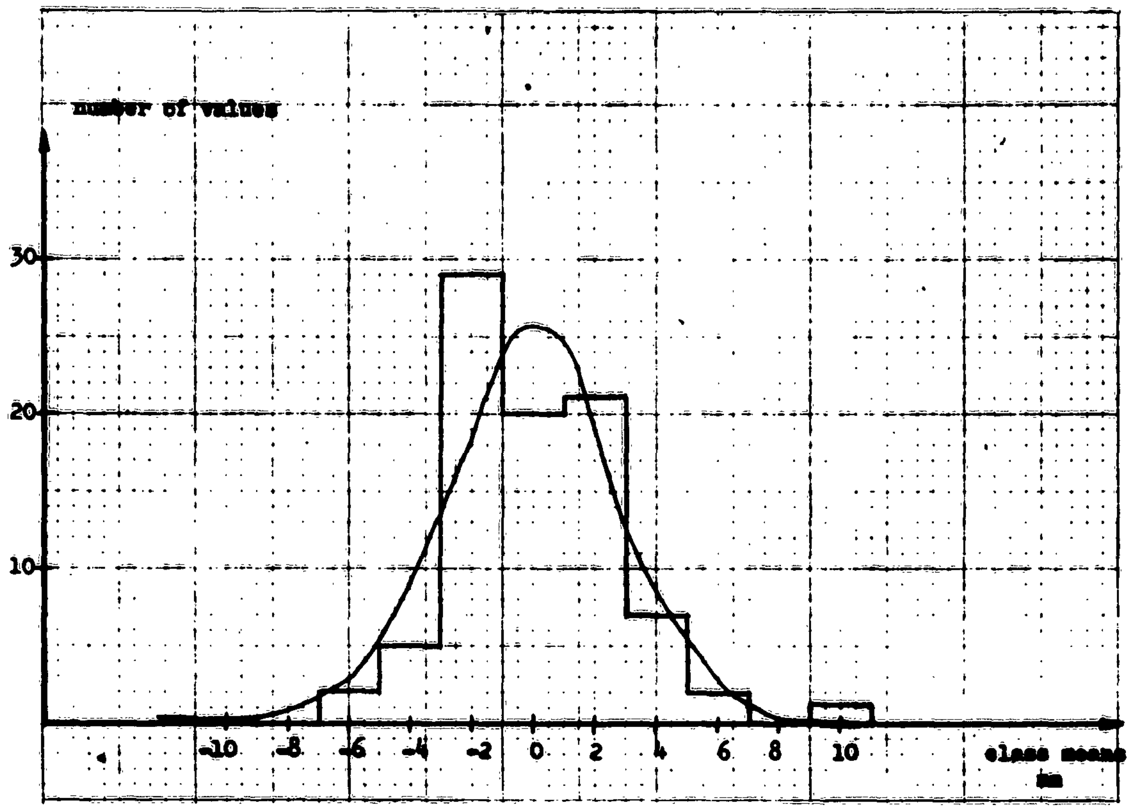
$chi^2_{95} = 1.64$

8.93

$chi^2_5 = 12.59$

Note that if a value coincides with a class limit it is counted with half a unit to each of the two adjacent classes.

Hence, the normal distribution hypothesis is accepted.



Histogram and normal distribution curve from the differences between two independent coordinate determinations of a photogrammetric test field.

1 3-4-5 = 43,547 = ARMY = FORT BELVOIR, VA.

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DA Task 8T35-12-001-02 Unclassified Report

Paper covers a series of tests of basic geometric quality of photographs obtained from aerial camera calibration in a multicollimator and from aerial photography obtained under operational conditions. The elements of interior orientation and most important regular (systematic) errors of image coordinates are determined as parameters in least squares adjustments, and remaining irregular errors of image coordinates are estimated as standard errors of unit weight. In multicollimator tests, film and glass plate negatives were tested and compared concerning geometrical quality. A criterion for tangential distortion is developed, and some correlation effects between residual image coordinate errors is studied. A variation of geometrical quality (weight) of image coordinates was found, and some possible reasons for this variation are more closely investigated. The resolving power, thickness variations of films, and locations of developed grains within emulsion were studied. Some tests of diapositive printers were made. Paper concludes: (a) method of least squares proved to be of value for tests of basic geometrical quality; (b) after correction for certain regular errors, image coordinates from glass plates and from films proved to be approximately of same geometrical quality; (c) geometrical quality and resolving power were found to be correlated; (d) considerable variations of film thickness were found; (e) illumination intensity and exposure time seem to influence position of details in emulsion; and (f) basic geometrical quality of reduction printers and influence upon geometry from dodging procedures can be conveniently tested with derived methods.

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